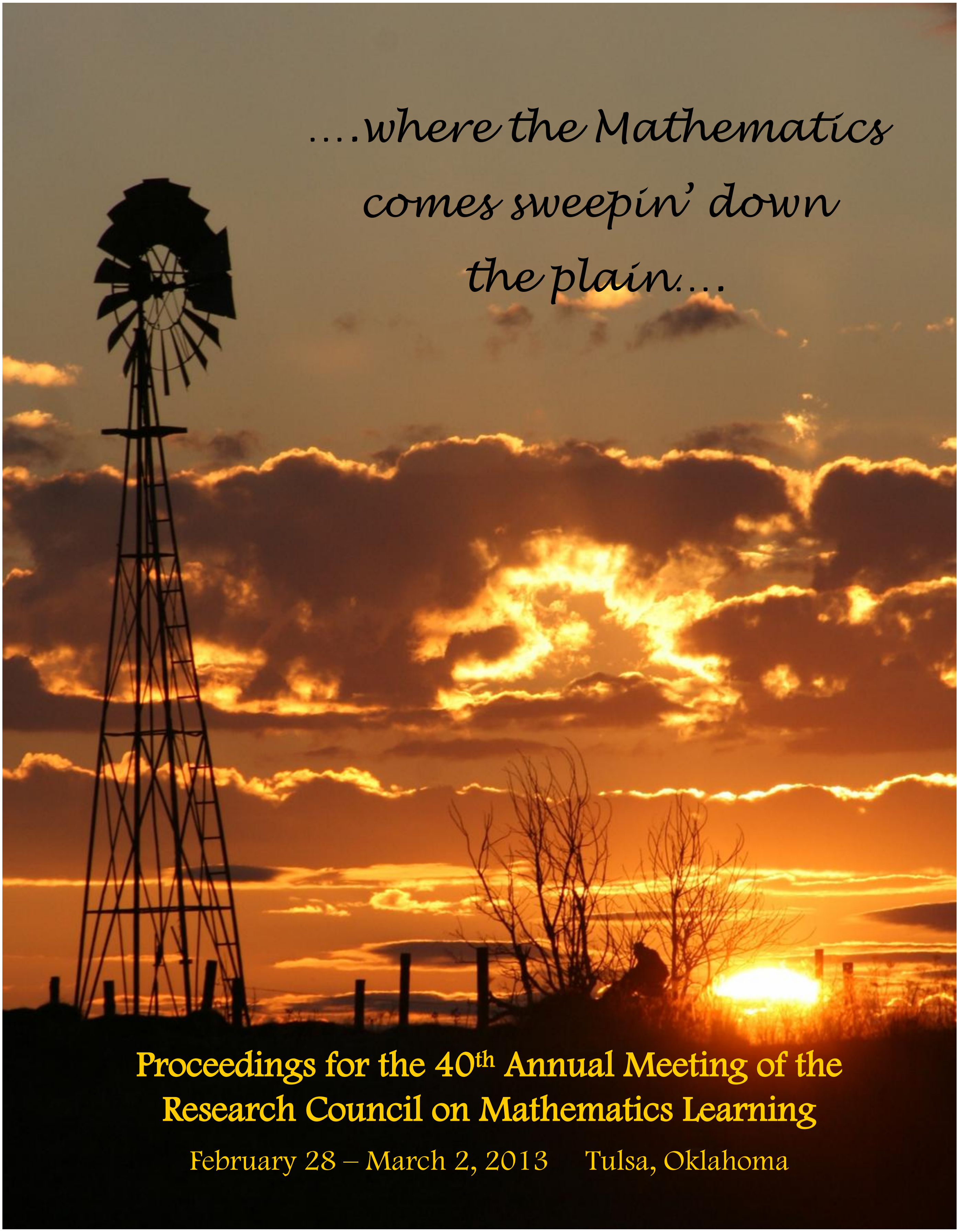

….where the Mathematics comes sweepin' down the plain….

Proceedings for the 40th Annual Meeting of the Research Council on Mathematics Learning

February 28 – March 2, 2013 Tulsa, Oklahoma

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RCML History

The Research Council on Mathematics Learning, formerly The Research Council for Diagnostic and Prescriptive Mathematics, grew from a seed planted at a 1974 national conference held at Kent State University. A need for an informational sharing structure in diagnostic, prescriptive, and remedial mathematics was identified by James W. Heddens. A group of invited professional educators convened to explore, discuss, and exchange ideas especially in regard to pupils having difficulty in learning mathematics. It was noted that there was considerable fragmentation and repetition of effort in research on learning deficiencies at all levels of student mathematical development. The discussions centered on how individuals could pool their talents, resources, and research efforts to help develop a body of knowledge. The intent was for teams of researchers to work together in collaborative research focused on solving student difficulties encountered in learning mathematics.

Specific areas identified were:

1. Synthesize innovative approaches.
2. Create insightful diagnostic instruments.
3. Create diagnostic techniques.
4. Develop new and interesting materials.
5. Examine research reporting strategies.

As a professional organization, the **Research Council on Mathematics Learning (RCML)** may be thought of as a vehicle to be used by its membership to accomplish specific goals. There is opportunity for everyone to actively participate in **RCML**. Indeed, such participation is mandatory if **RCML** is to continue to provide a forum for exploration, examination, and professional growth for mathematics educators at all levels.

The Founding Members of the Council are those individuals that presented papers at one of the first three National Remedial Mathematics Conferences held at Kent State University in 1974, 1975, and 1976.

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Teachers of Mathematics

Mathematics Learning

SECONDARY MATHEMATICS ALTERNATIVE CERTIFICATION TEACHER PROBLEM SOLVING

Brian R. Evans
Pace University
bevans@pace.edu

The purpose of this study was to understand alternative certification middle and high school teachers' mathematical problem-solving skills and perceptions. Participants were given a problem-solving examination and required to reflect upon their students' and their own problem solving. Findings revealed there was a significant improvement in problem-solving scores for the teachers over the course of the semester. Teachers perceived their students' problem-solving abilities as generally weak due to not understanding how to start a problem, lack of persistence, and poor literacy skills.

Problem solving continues to be of high importance in mathematics education (Posamentier & Krulik, 2008; Posamentier, Smith, & Stepelman, 2008) and is one of the five National Council of Teachers of Mathematics (NCTM) process standards (NCTM, 2000). National Council of Supervisors of Mathematics (NCSM) has considered problem solving to be the principal reason for studying mathematics (NCSM, 1978), and it has been recommended that mathematics content be taught from a problem-solving perspective (NCTM, 2000; Schoenfeld, 1985). Additionally, problem solving continues to be of high importance as an element of the *Common Core Standards* in which students must make sense of problems confronting them and persevere in solving the problems.

In order to understand what problem solving is, first the definition of a mathematical "problem" must be understood. Charles and Lester (1982) defined a mathematical problem as task in which (a) The person confronting it wants or needs to find a solution; (b) The person has no readily available procedure for finding the solution; and (c) The person must make an attempt to find a solution. According to Krulik and Rudnick (1989), problem solving is a process in which an individual uses previously acquired knowledge, skills, and understanding to satisfy the demands of an unfamiliar situation. Polya (1945), in his seminal work *How to Solve It*, outlined a general problem-solving strategy that consisted of (a) Understanding the problem; (b) Making a plan; (c) Carrying out the plan; and (d) Looking back.

The purpose of this study is to understand alternative certification middle and high school mathematics teachers' problem-solving abilities and perceptions about their students' and their own problem solving, which is critical in supporting them to teach from a problem-solving perspective. NCTM (2000) said, “Problem solving is not only a goal of learning mathematics but also a major means of doing so” (p. 52).

Theoretical Framework

In mathematics education, problem solving is the manifestation of constructivist learning, the theory that students learn best through constructing their own knowledge, as promoted by thinkers such as Jean Piaget and John Dewey. Authentic problem solving in mathematics is the basis of reform- and inquiry-based instruction in mathematics (Clark, 1997).

It has been shown that teacher beliefs about student ability greatly influence instructional practices (Nathan & Koedinger, 2000). Asman and Markkovits (2009) found teachers who are unable to solve difficult non-routine problems were less likely to include these types of problems on student assessments, even if they were willing to address such problems in their instruction. Rosales, Santiago, Chamoso, Munez, and Orrantia (2012) have noted that in the classroom problem solving can often take on a mechanized procedure in solving problems that involves limited situational knowledge. To counter this, Hobbs (2012) promoted situating problems using culturally relevant pedagogical techniques in order to help teachers better engage students from diverse backgrounds. Capraro, An, Ma, Rangel-Chavez, and Harbaugh (2012) advocated support for preservice problem solving and mathematics proficiency, particularly in open ended problem solving situations.

Polya (1945) laid the groundwork for systematic approaches to solving mathematical problems. Additionally, NCSM (1978) and NCTM (2000) have emphasized problem solving as the purpose of mathematics instruction and a way of teaching.

Research Questions

1. What differences were there in problem solving scores between the beginning and end of the semester in a mathematics content course for alternative certification teachers?
2. What were teacher perceptions of their students’ and their own problem-solving abilities? Further, what differences in perceptions of their student and their own problem-solving abilities existed between the beginning and end of the semester in a mathematics content course for alternative certification teachers

Methodology

The methodology of this study involved quantitative and qualitative methods. The sample consisted of 34 new teachers in the New York City Teaching Fellows alternative certification program enrolled in a graduate algebra content mathematics course for teachers that involved rigorous derivations and proofs. Teachers were given a problem-solving examination at the beginning and end of the semester. The problem solving examinations were different on the pretest and posttest instruments and the problems presented were unfamiliar to the teachers. The problems were selected from the literature. However, while the instruments were developed by an experienced mathematics educator with background in mathematics problem solving, a limitation is that the problem solving examinations were not analyzed for construct and content validity and reliability. Finally, teachers were also required to reflect upon both their students' and their own problem solving at the beginning and end of the semester.

Results

The first research question was answered using scores from the problem-solving examination, and data were analyzed using paired samples *t*-test (see Table 1), which revealed a statistically significant difference between pretest scores and posttest scores for the problem-solving examination, and there was a very large effect size. Caution should be taken in interpreting these results since it may be expected that teacher test scores would rise from pretest to posttest. However, the problems on the posttest were different problems from the pretest and were unfamiliar to the teachers taking the examination.

Table 1

Paired Samples t-Test Results for Problem Solving Ability

Problem Solving Examination	Mean	SD	*t*-value	*d*-value
Pretest	4.91	1.654	-8.679**	2.08
Posttest	8.35	1.649		

$N = 34$, $df = 33$, two-tailed

** $p < 0.01$

The second research question was answered using teacher reflections analyzed to determine teacher perceptions of student problem solving as well as their perceptions of their own problem solving. At the beginning of the semester teachers categorized their students as having weak problem-solving abilities and skills. The most commonly reported problems were knowing how to get started and persistence. Teachers said many students did not understand the problems they had to solve. At the end of the semester teachers found many of the problems they encountered with their students in the beginning of the semester still persisted. Teachers felt there were several things they could do to help improve their students' problem-solving abilities and skills. Most commonly mentioned were the steps to problem solving as outlined by Polya (1945). Teachers commonly said that scaffolding and differentiated instruction could be used to help improve problem solving in their students.

At the beginning of the semester teachers reported that they shared many of the issues that their students have such as knowing how to start, persistence, understanding what the problem is asking. At the end of the semester, most teachers said that having the algebra content class that focused on derivations and proofs had improved their problem-solving abilities greatly. Several used the phase, "I have come a long way," referring to their problem-solving abilities. Many said that it was the analytic nature of derivations and developing proofs that helped improve their problem-solving abilities. Additionally, many found understanding how mathematics "works" in the class furthered their analytic skills.

Conclusions and Educational Implications

Since there was an increase in problem-solving scores over the course of the semester it can be argued that the a strong mathematics requirement for alternative certification mathematics teachers, combined with their own teaching experiences, can lead to stronger problem-solving achievement, which is important given the emphasis of teaching mathematics from a problem-solving perspective (Clark, 1997; NCSM, 1978; NCTM, 2000; Posamentier et al., 2008). Future research should examine how much of this is due to the effects of content classes for teachers or how much is due to the effects of their teaching experience, particularly in alternative certification programs.

Teachers perceived that students did not persevere in their problem solving because they were reliant on the teacher giving them the solutions in previous years. While this reliance on teachers providing solutions may be partially due to negative attitudes toward problem solving held by the students (Arslan & Altun, 2007), it also could be a problem with teachers not giving enough time for students to engage in problem solving. Perhaps there is a need to give students more time in their problem solving, and to resist the temptation to simply "give" the solutions to the students. This should be further investigated.

One teacher said when he had time to work with one student individually he found great improvement in the student's problem-solving skills. Individual student attention is important to improving student learning (Foote, 2009; Himley & Carini, 2000). Future research should examine the impact of increased individualized attention on problem solving.

Strong problem-solving abilities and skills are essential not just in mathematics, but in other subject areas and life in general. It is important that teacher educators be aware of their pre- and in-service teachers' problem solving perceptions both for the students and the pre- and in-service teachers themselves. This is especially true for the many teachers who come to the profession through alternative pathways who increasingly teach in high-need urban schools. It is important that the students in high-needs schools receive the critical thinking and problem-solving preparation that they need for success in life.

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PROSPECTIVE TEACHERS' DEVELOPING CONCEPTIONS OF THE STANDARDS FOR MATHEMATICAL PRACTICE

Scott A. Courtney
Kent State University
scourtn5@kent.edu

The movement to adopt the Common Core State Standards for Mathematics impacts not only school districts and their teachers, but also university teacher preparation programs. In order to productively implement and sustain the Common Core's vision of developing mathematically competent students, preparation programs must support prospective teachers' development of practical conceptions of the Standards for Mathematical Practice. This article describes middle childhood (grades 4-9) pre-service teachers' engagements with activities designed to reveal their initial conceptions of the mathematical practices.

The national movement to adopt the Common Core State Standards for Mathematics has situated school districts and their teachers in positions primed for change and reform. Along with changes in mathematics content standards and their progressions, come increased emphasis on mathematical processes and proficiencies—the Standards for Mathematical Practice.

The two consortia awarded federal grants to design the Common Core assessment systems have indicated their respective assessments will include items and tasks requiring students to apply and connect mathematical content with the mathematical practices. For example, PARCC assessments will "include a mix of items, including short- and extended-response items, performance-based tasks, and technology-enhanced items (PARCC, 2012, pp. 4-5)… [that] will reveal students' content knowledge and elicit evidence of mathematical practices" (PARCC, 2012, p. 8). Therefore, providing K-12 students with opportunities to not only engage in problems, tasks, and activities that coherently connect content with the mathematical practices, but also experiences at exhibiting evidence of such knowledge and habits of mind in their written work, will become increasingly important as the Common Core assessments commence.

Literature Review

Although research regarding the mathematical practices is in its infancy, there exists a body of research pertaining to those processes and proficiencies that ground them. There is a growing body of research (e.g., Riordan & Noyce, 2001; Senk & Thompson, 2003) indicating that students in classrooms that utilize reform curricula (e.g., aligned to NCTM standards) not only perform significantly better on standardized achievement tests than do their counterparts in more traditional mathematics programs, but also outperform these same students on tests measuring conceptual understanding, applications, and problem solving ability. Such results suggest curricula that focus on the development of powerful processes and proficiencies can positively impact student achievement. Research also highlights teachers' difficulties in conceptualizing and providing students with opportunities to engage in these same processes and proficiencies (e.g., Jacobs et al., 2006).

Transition to the Common Core affects not only K-12 instruction, but also university teacher preparation programs—programs that will produce the next generation of teachers charged with enacting and sustaining Common Core's vision in their (future) classrooms. Such programs must provide prospective teachers with opportunities to experience, develop, and implement instruction and assessments meeting the demands of the Common Core, and opportunities to reflect on the impact of such instruction on their own and their (future) students' learning.

The current report adds to emerging research into teachers' conceptions of the mathematical practices by exploring the following research question: How do prospective middle childhood (grades 4-9) mathematics teachers (henceforth referred to as PSTs) conceptualize exhibiting engagement in the mathematical practices in written work?

Methodology

As part of a recent mathematics methods course, I required PSTs to solve mathematics problems (via "problem sets") related to the six domains of the grades 6-8 content standards (e.g., The Number System). Furthermore, PSTs were requested to solve the problems in a manner they believed would exhibit engagement in the mathematical practices in their written work. The majority of the problems were chosen from standards-based (i.e., reform) sources, such as the *Connected Mathematics Projec*t.

The course consisted of 16 PSTs and was the second of two math methods courses in PSTs' licensure program (grades 4-9). Data for this report pertains to Problems Sets #4 (domain: Statistics and Probability) and #5 (Geometry), and consisted of PSTs' written solutions, PSTs' choices for which mathematical practice(s) they believed they exhibited engagement in, and what PSTs took as evidence that any given practice had been engaged in. At the time PSTs were given the problem sets, their main experiences with the mathematical practices (in relation to the course) had involved supporting their images of what engagement in the practices looks like during verbal classroom interactions. Such support included viewing and discussing video from the Inside Mathematics website. Furthermore, due to the timing of PSTs' field experience (a course component), there were no in-class discussions of either problem set.

Analysis was both quantitative and qualitative. Quantitative analysis consisted of summary statistics and focused on the frequencies with which specific practices or practice combinations were chosen by PSTs amongst a problem, a domain, particular problem characteristics, or by a particular PST. Qualitative analysis involved the examination of PSTs' written descriptions for what they took as evidence that any given practice had been engaged in. Such analysis attempted to identify and characterize those mathematical practice aspects that appeared to be most influential in PSTs' identification of any given practice.

Findings

For Problem Set #4, only 12 PSTs completed the part of the assignment requesting they solve the problem and identify the mathematical practices (MPs) they believed students would engage in and potentially exhibit in their written work. One additional PST completed this part of the assignment for Problem Set #5. Furthermore, PSTs were asked to solve the problem and to think about how students might engage in the problem, prior to or in concert with making their practice selection(s).

Tables 1 and 2 display those practices PSTs identified for each of the seven problems of Problem Set #4 and #5, respectively. Specifically, the tables indicate PST by name, problem number (e.g., P1 is the first problem), and the mathematical practice(s) chosen (e.g., Amie indicated problem #1 of Problem Set #4 involved MP.1 and MP.6). "None" indicates no practices were identified for that problem.

Table 1

Identified Mathematical Practices by PST and by Problem (Problem Set #4)

PST	P1	P2	P3	P4	P5	P6	P7
Alejandra	7	1	8	3	1	1, 3	1, 3, 4, 5
Amie	1, 6	7	2	2	none	1, 2	4
Blondell	4, 8	2, 6	1, 6	7	1	2	8
Bulah	2, 4, 5, 6	1, 5, 8	1, 5, 6	3, 5	1, 3, 4, 6, 7, 8	2, 3, 4, 7	4
Jamie	1, 8	1, 5	6, 8	1, 8	1, 5	none	1, 4, 5, 6
Kelly	1, 3, 6	1, 4, 5	1, 3, 4, 5	1, 4, 5, 6	1, 6	1, 3, 4, 6	1, 8
Kurt	2, 4, 7	5, 6	2, 3	1, 4, 7	2, 4, 5, 7, 8	1, 8	1, 4, 6
Loraine	2	1	7	6	1	3	1, 4, 6, 8
Myra	1, 3, 4, 5	1, 2, 3, 5	1, 2, 3, 5	1, 2, 3, 5	1, 3, 4, 5	1, 3, 4, 5	4
Neil	1, 3, 4, 8	1, 4, 5	1, 4, 6, 8	1, 5, 6	1, 5, 6	1, 6	1, 4, 5, 6
Stella	1, 2, 5, 6	1, 3, 4, 5, 6, 7	1, 3, 7	1, 2, 4, 5	1, 2, 5, 7	none	none
Valene	1, 4, 5, 6, 7, 8	1, 4, 5, 6, 7, 8	1, 4, 5, 7	1, 4, 5, 7	1, 4, 5, 7	1, 4, 5, 7	1, 5, 7, 8

Table 2

Identified Mathematical Practices by PST and by Problem (Problem Set #5)

PST	P1	P2	P3	P4	P5	P6	P7
Alejandra	1, 4	4	3, 7	3, 7	8	1, 3	3
Amie	none	1	2	6	3	4	none
Blondell	3, 7	4	4	4	2	4	4
Bulah	1, 4, 5	2, 4, 5, 8	1, 2, 5, 7	2, 3, 5, 7	1, 3, 8	1, 4, 5, 6	1, 4, 5, 6
Carlene	1, 4, 7	1, 4, 6	2, 4	1, 3	3	2, 4	2, 4, 7
Jamie	2, 4	1	6, 8	1, 7	1, 7	1, 4, 5	1, 4, 5
Kelly	1, 2, 3, 4, 6	1, 3, 5, 6	1, 2, 3, 6	2, 3	1, 2, 3	1, 2, 3, 4, 5, 6	1, 4, 5, 6
Kurt	1, 4, 6 7	1, 4, 7	6, 7	4, 6, 7, 8	6, 7	1, 2, 7	1, 3
Loraine	1, 2	4	3	1, 3	3, 8	1, 2	1, 4
Myra	1, 3, 4, 5, 6	1, 3, 4, 5	1, 2, 3	1, 3, 4, 5, 6	1, 2, 3, 6	1, 3, 4, 5, 6	1, 3, 4, 5
Neil	1, 4, 6	1, 4, 5, 6	1, 4, 5, 6	1, 3, 4, 5	1, 3, 5, 6, 8	1, 3, 6	1, 5, 6
Stella	1, 2, 3	1, 2, 5, 6	1, 3, 7	1, 2, 3, 6, 7	1, 2, 7	1, 4	1, 2, 3
Valene	1, 4, 5, 6	1, 4, 5, 6, 7	1, 4, 5, 6	1, 4, 5, 6, 7	1, 5, 6, 8	1, 4, 5, 6	1, 2, 4, 5

As illustrated in the tables above, there was a reasonable degree of variability in the mathematical practices (MPs) chosen amongst and within problems, and amongst and within PSTs for each problem set. In addition, there was a reasonable degree of variability amongst the combinations and number of practices chosen. For example, for problem #2 of Problem Set #4 (Table 1), the number of practices chosen by any one PST ranged from one (Amie) to six (Stella). Furthermore, although Bulah and Blondell each solved problem #1 of Problem Set #5 showing very similar written work, Bulah identified the problem as involving MP.3 and MP.7, whereas Blondell identified MP.1, MP.4, and MP.5.

Although such results might be expected, considering the potential for idiosyncratic interpretations of the mathematical practices, the interaction and overlap amongst practices (PARCC, 2012, p.13), and the limited opportunities PSTs had to discuss and operationalize the practices, my intent was to gather data with which to develop a baseline for PSTs' conceptions of the mathematical practices. Such a baseline would then serve to guide future engagements with these and other teachers.

A particular interesting result involved the frequency with which certain pairs of mathematical practices were identified. Figure 1 illustrates the frequency with which pairs of practices were chosen by PSTs (weight of pair connection) for the two problem sets combined. For example, MP.4 and MP.5 occur together in each of the combinations 1, 4, 5 and 4, 5, 7, 8.

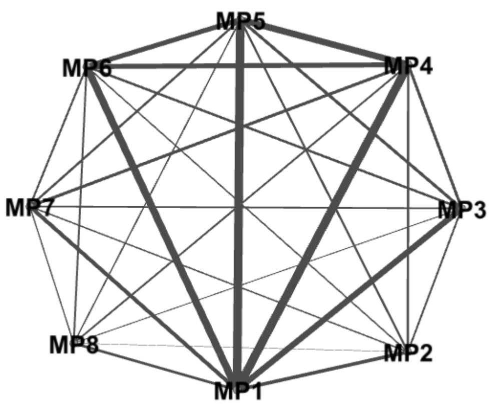

Figure 1. Strength of mathematical practice pairs. This network graph displays the strength of pairs of mathematical practices for Problems Sets #4 and #5 combined (generated with Gephi, www.gephi.org).

As Figure 1 indicates, other than pairs that included MP.1 (e.g., MP.1 and MP.5), the pairs MP.4 and MP.5, MP.4 and MP.6, and MP.5 and MP.6 occurred with the greatest frequency. This result also held true when each problem set was examined individually. The frequency with which the pair MP.4 and MP.5 occurred might be accounted for in light of their relationship (modeling and using tools) in McCallum's (2011) higher order structure to the practice standards.

In order to attempt to explain the frequencies with which individual, pairs, or combinations of practices were chosen by PSTs, I looked for potential relationships between the problems' features and the practices selected. Particular features included: whether the problem asked for an explanation, involved a realistic context, asked students to critique another's reasoning or justify their statements, included a mathematical representation or object (e.g., diagram, table, graph, formula, triangle), or requested a mathematical representation be constructed. The main reason for focusing on the problems' features was due to PSTs' limited experiences with the practices. As such, I anticipated much of PSTs' decision making would be based on what they deemed as relevant between the problems' features and the practice descriptions provided in the Common Core documents. Table 3 displays the problem features, the problems associated with each feature (PS4: P6 indicates problem #6 from Problem Set #4), and the standard score for each mathematical practice (MP).

Table 3

Mathematical Practice Standard Score by Problem Feature

	Standard Score (z)				
	Explain PS4: P1, P3, P4 PS5: P1, P2, P3, P4, P6	Context PS4: P1, P2, P4, P6, P7 PS5: P1, P6, P7	Critique/Justify PS4: P6 PS5: P4, P5	Includes Rep PS4: P1, P2, P3, P4, P5 PS5: P3, P4, P5, P7	Requests Rep PS4: P7 PS5: P1, P2
MP.1	1.78	1.80	1.89	2.12	1.50
MP.2	-0.44	-0.56	-0.13	-0.57	-0.79
MP.3	-0.13	-0.39	0.88	0.03	-0.67
MP.4	0.89	1.04	0.38	0.03	1.50
MP.5	0	0.27	-0.63	0.37	0.01
MP.6	0.13	-0.12	-0.63	-0.10	0.13
MP.7	-0.70	-0.88	-0.63	-0.57	-0.79
MP.8	-1.52	-1.16	-1.14	-1.31	-0.90

As Table 3 illustrates, problems involving a realistic context were associated with MP.4 being selected (z = 1.04). This was anticipated considering MP.4's description, "Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace" (CCSSI, 2010, p. 7). For Problem Set #4, MP.5 was also frequently chosen (z = 0.51) for "Context" problems, but not for Problem Set #5 (z = 0.03). Problems explicitly requiring students to critique or justify were associated with MP.3 being selected (z = 0.88). This result was also expected due to the nature of MP.3—"Construct viable arguments and critique the reasoning of other" (CCSSI, 2010, p. 6). Problems requiring students to explain their work, their thinking, or their reasoning were associated with MP.4 being selected (z = 0.89). This result was surprising, since I anticipated such problems would motivate PSTs to choose MP.3, "construct viable arguments" (CCSSI, 2010, p.6) and/or MP.6, "communicate precisely to others" (CCSSI, 2010, p. 7). The standard score for MP.4 was larger for such "Explain" problems in Problem Set #5 ($z = 1.08$) than for Problem Set #4 ($z = 0.21$).

Problems including a mathematical representation were only associated with MP.1. For Problem Set #4 alone, including a representation (e.g., table, graph) was associated with MP.5 ($z = 0.80$), suggesting PSTs conceived such representations as being tools. Alternatively, for Problem Set #5, including a representation or object (e.g., a triangle), even for problems not also asking for an explanation, critique, or justification, was associated with MP.3 ($z = 1.01$). Finally, problems requesting students construct a mathematical representation (e.g., table, chart) were associated with MP.4 ($z = 1.50$), suggesting PSTs might have focused on MP.4's statement, "Mathematically proficient students…are able to…map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas" (CCSSI, 2010, p. 7).

PSTs were also required to articulate where and how they believed their written response exhibited the chosen practice(s) being employed. Unfortunately, only five PSTs responded to this request for Problem Set #4, and only four of those responded for Problem Set #5.

For PSTs, engagement in MP.1 was associated with employing or using a given or created mathematical representation as part of the solution process. Specific instances included: "When I created tree diagram to help me come up with the different combinations" (Neil) and, "Occurs by understanding the box plot info and using it to solve the problem"(Jamie). Myra, who chose MP.1 for all 14 problems, indicated, "The first thing…all students have to do…is to make sense of problems and persevere in solving them. If a student cannot do this they have little to no chance of solving the problem."

Engagement in MP.4 was associated with creating and interpreting some form of mathematical representation. Specific instances included: "Student must make a tree diagram to find all possible combinations" (Alejandra) and, "When I drew my hexagons to explain my answers" (Neil). Engagement in MP.5 was also associated with using or interpreting a mathematical representation, which helps to explain the frequency with which these practices were chosen in concert. Specific instances included: "When I used the graph to conclude my answers (Jamie)" and, "Read and manipulate grid and picture to help you solve the problem" (Myra). For Neil, the use of paper and pencil to draw geometric objects (e.g., triangles), as part of the solution process, was indicative if engaging in MP.5.

Finally, engagement in MP.6 was associated with the use of mathematical definitions, terms and symbols, and with working with units. Specific instances included: “When I used clear definitions” (Neil), “When dealing with units” (Amie), and, “Figure out what the “O” really means and explain what the origin tells us about the triangle (Myra).

Discussion

Although the mathematical practices that students have the potential to engage in depends on both the cognitive demand of the problem, task, or activity and its implementation, a teacher’s (and their students’) conceptions of the practices also play a significant role. PSTs in this study demonstrated restricted meanings for the practices—focusing on connected, but limited components of the practice descriptions. Specifically, although having a realistic context initially appeared to influence PSTs’ choice of MP.4, PSTs’ descriptions suggest a focus on the creation and interpretation of mathematical representations (i.e., tools or models). PSTs’ descriptions for MP.5 suggest a focus on whether or not a mathematical representation was used in the problem solving process. In addition, although some PSTs associated the use of mathematical representations as a means to make sense of the problem (MP.1), Myra indicated that engagement with MP.1 occurred almost by default (as long as the problem was able to be solved). Future research must explore how to support teachers’ development of mathematical practice conceptions of sufficient robustness to manage the development of similar increasingly sophisticated habits of mind in their students. Furthermore, larger scaled studies examining teachers’ conceptions of engagement in and exhibition of the mathematical practices over an increased sample of problems covering each relevant domain would help identify those practices or practice components that are the most difficult for teachers to operationalize, and provide insight into how best to support teachers in enacting and sustaining Common Core’s vision of developing mathematically competent students.

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PRESERVICE TEACHERS' EMOTIONAL ENGAGEMENT WITH THE TOWER OF HANOI

Elaine Young
Texas A&M University-Corpus Christi
Elaine.Young@Tamucc.Edu

Sarah Ives
Texas A&M University-Corpus Christi
Sarah.Ives@Tamucc.Edu

Jose Guardiola
Texas A&M University-Corpus Christi
Jose.Guardiola@Tamucc.Edu

The Tower of Hanoi is a traditional problem solving task for several fields of discipline. Preservice teachers engaged with this task and provided reflective journals and questionnaires which were used to answer two research questions: "What emotions do preservice teachers experience before, during, and after the task?" and "How do emotions change across time?" Journals were mined for emotional words/phrases; questionnaires were scored on a Likert scale and then analyzed. Results indicated significant quadratic behavior in emotions over time. Student voices in journals supported outcomes of qualitative and quantitative analyses.

> *If the student had no opportunity in school to familiarize himself with the varying emotions of the struggle for the solution, his mathematical education failed in the most vital point (Polya, 1957, p. 94).*

The Tower of Hanoi puzzle has its roots in Cardano's description of the Far East (Danesi, 2004). In 1883, Edouard Lucas, a French number theorist, marketed the puzzle as a brain teaser (Poole, 1994). The puzzle is a popular task in mathematics and a traditional programming problem in computer science. Kopecky, Chang, Klorman, Thatcher and Borgstedt (2005) described its use in psychology and psychiatry to test for mental disorders, attention measures and problem-solving efficacy.

The Tower of Hanoi task was used in the first of three mathematics content courses for preservice elementary and middle school teachers. The task paired accessibility with high cognitive demand, as well as a blend of challenge and intrigue. It encouraged students to collaborate, communicate mathematically, and develop mathematical reasoning. The task models Bruner's (1966) successive modes of intelligence: enactive, iconic, and symbolic representations, as students worked with concrete models, then recorded data, and finally identified recursive and explicit formulas. Students wrote reflective journals; instructors noticed that there were emotional words or phrases indicating positive and negative feelings. A questionnaire was then added, which was completed three times during the task: before, during, and after.

The purpose of this study was to examine the emotions reported by preservice teachers while engaged with the Tower of Hanoi task. Qualitative and quantitative data were gathered from reflective journals and a questionnaire. Research questions included:

1. What emotions do preservice teachers experience before, during and after engaging with the Tower of Hanoi task?
2. How do emotions change across the problem solving process?

Related Literature

Tower of Hanoi Task

Anderson, Albert and Fincham (2005) used the Tower of Hanoi task to identify brain regions used when problem solving. The task uses different cognitive and motor actions in rapid succession. By tracking the patterns of these actions across brain regions, they could predict when participants were planning future moves.

Mau and D'Ambrosio (2003) used the Tower of Hanoi task with preservice elementary teachers, who wrote reflections on their experiences. Students shared inner tensions as they tried to make sense of their own and others' thinking. They described their insights in learning and expressed increased interest in mathematics.

Emotion and Problem Solving

Emotion research began in the 1980s, showing that emotions have their own memory pathways and serve as a critical source of information for learning. Experiences that are laden with emotion are more easily recalled than neutral events (LeDoux, 1994), and fMRI scanning showed that emotional events are more likely to be retained (Dolcos, LaBar & Cabeza, 2004). Duvallet and Clement (2005) identified emotional manifestations in complex cognitive activities by recording facial expressions and the degree and amplitude of skin conductance. They found that emotions were observed more often when the subject was stuck or blocked than during the exploratory phase. They concluded that emotions may guide the activity, especially when making decisions.

Belavkin (2001) claimed that emotion always accompanies and makes a positive contribution to the process of problem solving. Friesen and Francis-Poscente (2009) proposed that "experiences that are charged with mathematical emotion are not some kind of extraneous distraction or curious side effect, but they are at the very core of involvement with mathematics" (p. 166). Appropriate emotional conflict can positively affect psychological and learning functions; emotions are not always hostile factors to be eradicated from the learning process. Allowing both positive and negative emotions in problem solving is not necessarily demeaning or detrimental (Allen & Carifio, 2007). In fact, Thompson and Thompson (1989) claim that frustration is required in developing an appreciation for problem solving. Belavkin (2001) found that positive emotions during problem solving were accompanied by increased motivation and confidence.

Despite a common belief that emotion during problem solving can be disruptive, distracting, and diminish performance, emotions experienced during problem solving have been found to energize, organize, focus, and improve performance (Allen & Carifio, 2007). A challenging problem coupled with success in finding a solution can inject positive feelings into schemas in terms of success, rewards, satisfaction and competence (Allen & Carifio, 1995).

Methodology

Setting and Participants

The Tower of Hanoi task was purposefully planned for the first week of class and was intended to set expectations for the course, including collaborative problem solving, communication and justification. Students worked in table groups with manipulatives to model the task, recorded the minimum number of moves on a table. They then worked together to identify the recursive and explicit formulas for *n* disks.

Students (N = 275) were enrolled in the first of three content courses for elementary and middle school preservice teachers at a mid-sized university in the southern U.S. The greatest majority of students were female; approximately half were White and half were Hispanic; 80% were seeking elementary certification.

Instruments and Analyses

Students wrote three-page reflective journals describing their engagement with the Tower of Hanoi task -- how they thought about it, what they tried, methods they used, and what they found in the end. These qualitative data were collected over nine semesters from 2004-2011. The researchers 1) read each journal and identified emotional words or phrases, 2) grouped similar words or phrases, for example: *frustrate*, *frustrated*, and *frustrating*, 3) searched each stem (such as “frustrat*”) across journals and 4) recorded a total count of occurrence for each word/phrase.

Allen and Carifio (1999) developed the Emotion Questionnaire as a 38-item instrument used to evaluate various aspects of mathematical problem solving. The questionnaire measures emotion (and four other traits) using category subscales with highly reliable internal consistency estimates. This study focuses on the Emotional Activity category. Items within a category are scattered throughout the questionnaire; some items are reversed. Each Likert-scale item is scored from 1 to 6, with a higher score indicating a higher level of positive emotion (Carifio, 2004).

For this study, the researchers adjusted the questionnaire to six spaces for each sample item, eliminating a center value (see Figure 1). Students completed the questionnaire three times during class: after reading the problem but before attempting the task, during problem solving, and upon completion of the task or the end of class. The researchers analyzed the Emotion Activity scores across time using repeated measures.

	Emotion Activity	
Distressed	___:___:___:___:___:___	Delighted
Good	___:___:___:___:___:___	Bad
Successful	___:___:___:___:___:___	Unsuccessful
Frustrated	___:___:___:___:___:___	Satisfied
Proud	___:___:___:___:___:___	Ashamed
Pleasant	___:___:___:___:___:___	Unpleasant
Annoying	___:___:___:___:___:___	Pleasing

Figure 1. The paired items for Emotion Activity (adapted from Allen & Carifio, 1999).

Findings

Qualitative Data

Data consisted of 275 student reflections and 101 questionnaires. Analysis of the journals returned a total of 116 words/phrases that indicated positive, negative or neutral emotions. Fourteen of the words/phrases occurred in more than 35 journals (see Figure 2).

Emotion word or phrase	Number of journals containing word/phrase (N = 275)
Difficult	118
Easy	112
frustrate; frustrated; frustrating	99
Hard	96
challenge; challenging	87
Simple	78
Fun	74
Interesting	66
confused; confusing; confusion	59
succeed; succeeded; success	54
excited; exciting	50
Enjoy	47
got stuck	38
accomplish; accomplished; accomplishment	37

Figure 2. Emotion words/phrases mentioned in more than 35 journals.

There were indications of the pairing of positive and negative emotions in student journals. A student wrote, "It's interesting to me because no one can really understand the wonderful feeling of figuring something out until they have really been frustrated with the problem." Some students commented on the questionnaire in their reflective journals. A student wrote, "I also liked the paper we filled out asking us about how we felt and stuff because it showed me how my mood had changed throughout the activity from the beginning, in the middle, and at the end. I noticed a big difference from my middle mood to the beginning and end mood."

Quantitative Data

A general linear model was set up with the three repeated emotion measures as the dependent variable. Individuals were considered as blocks; the variables gender, class grade, GPA and degree program were independent variables, none of which were found statistically significant and were dropped from the model. The analysis continued using a repeated measures procedure, resulting in a statistically significant difference across emotions for the before, during and after measurements ($F = 11.13$, $p < 0.0001$). Results also showed a statistically significant quadratic behavior using an appropriate contrast ($F = 14.87$, $p = 0.0001$). Additionally, using a HSD Tukey's analysis procedure, emotion score means after treatment were found significantly higher from means during ($p < 0.0001$) and before ($p = 0.0185$) treatment.

Quadratic behavior over time was also indicated in student journals. A student wrote, "The Tower of Hanoi was very challenging and frustrating at first. Then in the middle of trying to figure out the puzzle I had a breakthrough and everything started to come together and I wasn't so frustrated. When I was able to solve the puzzle and figure out the pattern to solve it I felt relaxed and very pleased that I was able to come up with the answer to the puzzle."

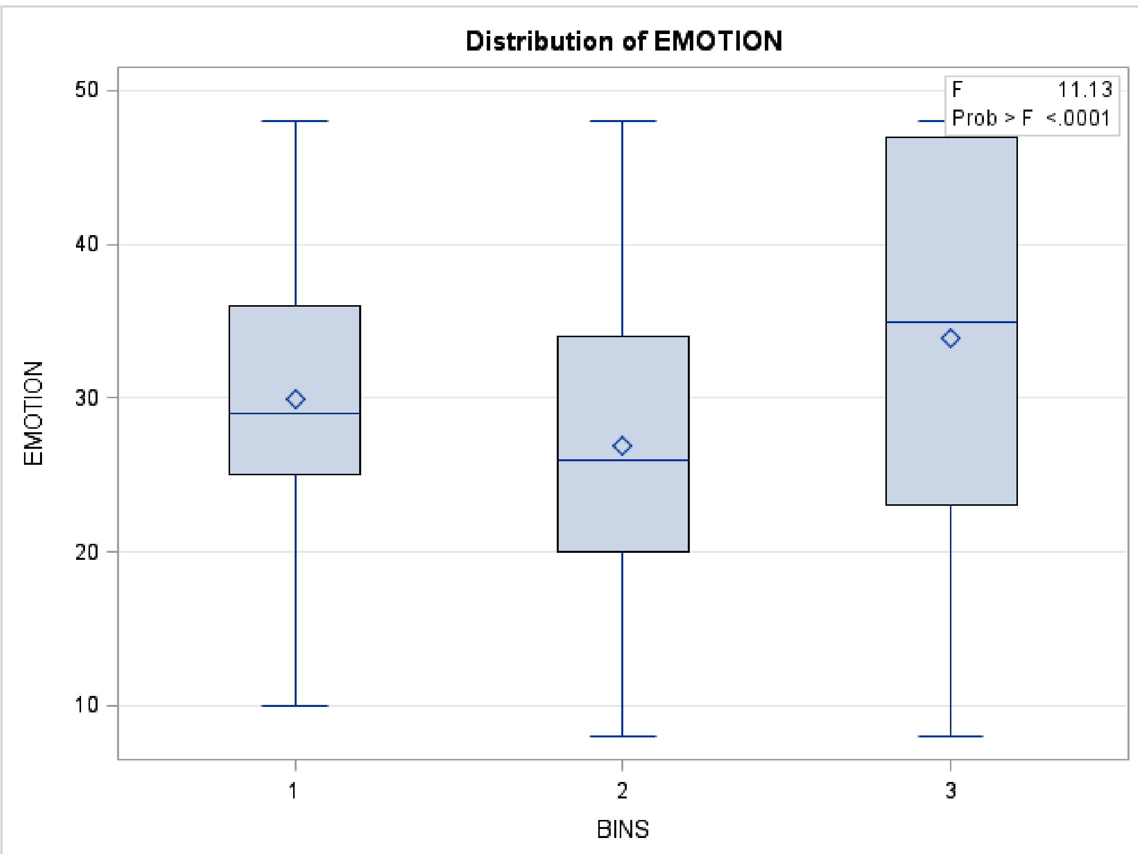

Figure 3. Distribution of emotions before, during and after the task (Bins signify scored questionnaires before, during and after engaging with the task).

Conclusions

Struggle and frustration during the task were balanced mostly by positive emotions at the outcome, such as accomplishment and satisfaction. A student wrote, "I was definitely frustrated again…. I was so excited, way too excited for just having completed a math problem but I sure felt like I accomplished something." Another explained, "Even though it took a while for us to solve it I felt really proud of myself afterwards. I felt like I had accomplished something that not most people could." Results from this study were similar to those found in the literature, that challenge and struggle can be paired with satisfaction and accomplishment. When preservice teachers are engaged with such tasks and find the value of working through frustration to elation, it may help change their beliefs and attitudes toward mathematics.

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Mathematics Teacher Candidates' Understanding of Function

Stacy Reeder
University of Oklahoma
reeder@ou.edu

Rachel Bates
Redlands Community College
rachel.m.bates-1@ou.edu

Functions play in integral and important role throughout mathematics. Many studies have focused on college student understanding of function and have found them to be lacking. Few studies however, have focused on teacher candidates' understanding of function. Given that middle school and high school mathematics teachers help students develop what we hope will be a deep and flexible understanding of function it is important that their own understanding of function be rich and well-developed. This study examined one group of mathematics teacher candidates' understanding of function. The results indicate that their understanding was limited.

The idea of a function, or at the very least, the anticipation of the idea of function can be dated back as far 2000 B.C.E. and is evidenced in the work of the Babylonians and ancient Greeks with one-to-one correspondence for counting and their extensive use of tables. However, the notion of function, as we know it, did not arrive on the mathematics landscape until the early 1300's and had its beginnings as a way of designating the correspondences between geometrical entities. Over time the notion of function continued to develop and become associated with the study of analytical expressions, thus securing a central place in mathematics (Burnett-Bradshaw, 2007).

The work of Oreseme (1323 – 1382) included "general ideas about independent and dependent variable quantities seem to be present" (Ponte, 1992, p. 4). Some two hundred years later Descarte (1596 – 1650) indicated a dependence between variable quantities in his work with equations in two variables marking the emergence of the notion of functions as an individualized mathematical entity. Furthering the idea of function was Newton (1642 – 1727) who demonstrated how functions could be developed in infinite power series. While Leibnitz was the first to use the term "function" in 1673, the study of function as a clearly individualized concept did not arise for a few more decades at the end of the 17th century. Finally, as a result of "the development of the study of curves by algebraic methods, a term to represent quantities that were dependent on one variable by means of an analytical expression was increasingly necessary" (Ponte, 1992, p.4), the term "function" was adopted. This was decided somewhere between 1694 and 1698 in an exchange between Leibnitz (1646 – 1716) and Bernoulli (1667 – 1748).

Related Literature

Since the time "function" was established as an individualized concept in mathematics it has played an important role throughout the mathematics curriculum. Cooney, Beckmann, and Lloyd (2010) purport that "functions compose a major area of school mathematics that is crucial for students to learn but challenging for teachers to teach" (p. 1). They continue by stating:

> Learners often have a narrow view of functions. On the basis of their frequent use of linear and quadratic function, students tend to limit the concept of functions to equations or orderly rules. They frequently overlook many-t-one correspondences or irregular functions that could be very useful in describing real-world phenomena (p. 1).

Understanding that "the concept of function is central to students' ability to describe relationships of change between variables, explain parameter changes, and interpret and analyze graphs" (Clements, 2001, p. 745) is supported by the National Council of Teachers of Mathematics. They advocate in their Principles and Standards for School Mathematics (NCTM 2000, p. 296) that instruction across all grade levels should "enable all students to understand patterns, relations, and functions" (p. 296). Unfortunately, research studies indicate that although, the concept of function is important in mathematics and should be developed over many years with students, high school and college students have difficulty understanding function with any depth and flexibility.

Studies conducted in the 70's and 80's well documented students' difficulties with understanding function noting, among other things, that their mathematical understanding related to function often involved incorrect ideas and that the understanding they had developed was often narrow in scope (See Leinhardt, Zaslvasky, & Stein (1990) for a survey of the literature). The reasons cited for these challenges are plentiful and well documented, frequently pointing to the notion that how students are introduced to functions and the problems they are asked to solve related to functions are limiting (Buck, 1970; Dreyfus & Eisenberg 1982; Freudenthal,1982; Herscovics, 1982; Kaput, 1987; Lovell, 1971; Orton, 1970; Sierpinska 1992; Tall 1996; Vinner & Dreyfus, 1989). For example, a more recent study conducted by Clements (2001) included thirty-five high school pre-calculus students. She found that only four could provide a definition that was consistent with or similar to the mathematical definition which includes the idea that every element in the domain must be mapped to a unique element in the range. Further, students who participated in this study seemed to focus primarily on graphical representations of the function and applied the vertical line test in order to determine if a graph represented a function or not.

Since research has documented that mathematics students in both high school and college-level settings have limited understandings of function this raises questions about the developed understanding of function of our high school mathematics teachers and high school mathematics teacher candidates. Wilson (1994) examined the understanding of functions of one secondary mathematics teacher candidate and found her initial understandings were primarily computational (function machines, point plotting, vertical line test) and were in line with her predominant view of mathematics as a collection of procedures. Further, Even (1993) found that the prospective teachers in his study had a limited conception of function and it influenced their pedagogical thinking about functions. The implication of his study was that with only a limited conception of function themselves, these teachers would have no choice but to provide their own students with rules to follow for functions without concern for understanding. While there have been studies focused on mathematics teacher candidates understanding of functions, those particularly focused on a multiple representations prospective of functions are not well documented.

The National Council of Teachers of Mathematics has, for several decades, supported the development of a rich and flexible understanding of functions that includes student work with functions through multiple representations (NCTM, 1989; NCTM, 2000; Cooney, et al., 2010). In light of this emphasis and the adoption of the Common Core State Standards for Mathematics (2010) which includes functions as a primary strand for secondary mathematics students, this study aimed to investigate one group of secondary mathematics teacher candidates' understandings of function. Thus, the research question was as follows:

What are secondary mathematics teacher candidates' understandings of function?

Methodology

This study, involving one class of secondary mathematics teacher candidates (N=7), employed qualitative research methods to examine the research question and analyze the data. A twenty-three item pre/post-test was developed using items from several sources (see list following the References) and was constructed such that participants were asked to identify whether what was presented was a function and to explain their reasoning. Functions were presented on the pre/post-test in multiple representations including, graphical, verbal, pictorial, and tabular. Each type of representation was considered separate from one another for analysis and scored using the following rubric:

Score	Criteria
0	incorrect answer, incorrect reasoning
1	correct answer, insufficient or incorrect reasoning
2	correct answer, correct reasoning

Both researchers scored the participants pre- and post-tests apart from one another and then met to reconcile any differences in scoring until 100% agreement was found.

In addition to the pre/post-test, participants were asked to respond to the following journal prompt that was included as part of the data for this study: What is a function? These data were analyzed using the categories provided by Schwingendorf, Hawks, and Beineke in their 1992 article focused on students' conception of function. Their categories suggest a hierarchical understanding of function ranging from what the authors term as "prefunction" to "dependence" as described below:

> ***Prefunction***: a response which appears to indicate little or no concept of function.
> ***Action***: a response which indicates a replacement of a number for a variable and then computing to obtain a number where there is no indication of an overall process of transforming a number to obtain another number.
> ***Process***: a response which indicates a coherent use of an input, a transformation, and an output in a general way.
> ***Correspondence***: a response which indicates a correspondence between two variables.
> ***Dependence***: a response which indicates a dependence between two variables.

The data produced in response to the "what is a function?" prompt were analyzed independently using the aforementioned categories by each researcher and then the results were compared in an effort to reach consensus.

Setting for the study

The participants in this study were senior mathematics teacher candidates enrolled in a methods course focused on the teaching Algebra and Geometry concepts. As such, they had completed three years of a rigorous four-year undergraduate program in mathematics education wherein fifty-one hours of their program coursework is mathematics. Most participants had completed more than thirty hours of college-level mathematics beyond Calculus I. During this semester, about half of the course readings and time in class were focused on the notion of functions from the perspective of two big ideas: multiple representations of functions and families of functions (Cooney, et al., 2010). The readings and activities for this course engaged the students in learning about not only the multiple representations of functions and families of functions (e.g., the concept of "parent functions" was a particular focus) as content knowledge for teaching but also the importance of teaching their future students with these big ideas about function as the focus.

Findings

The results of the pre- and post-test indicate little to no improvement in each of the 4 areas of focus and are shown in Tables 1-4 below.

Table 1

Graphical Representations

Item	1		2		3		4		5		6		7	
Score	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
0			17%	14%	33%	57%			17%	14%	17%	14%	17%	14%
1	33%	14%					17%	14%						
2	67%	86%	83%	86%	67%	43%	83%	86%	83%	86%	83%	86%	83%	86%

Item	8		9		10		11		12		13		Overall	
Score	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
0	17%	14%	33%	57%	17%	14%					17%		14%	15%
1							33%	14%	33%	14%			9%	4%
2	83%	86%	67%	43%	83%	86%	67%	86%	67%	86%	83%	100%	77%	81%

Table 2

Verbal

Item	14		15a		15b		15c		15d		Overall	
Score	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
0	33%	57%	17%				50%	29%	17%	14%	23%	20%
1			33%	14%	17%		17%		33%		20%	3%
2	67%	43%	50%	86%	83%	100%	33%	71%	50%	86%	57%	77%

Table 3

Pictorial Representation (Growth Patterns)

Item	16a		16b		Overall	
Score	Pre	Post	Pre	Post	Pre	Post
0	17%		17%		17%	
1	67%	86%	67%	57%	67%	71%
2	17%	14%	17%	43%	17%	29%

Table 4

Table Format

Item	16c		16d		16e		Overall	
Score	Pre	Post	Pre	Post	Pre	Post	Pre	Post
0			17%	14%			6%	5%
1	83%	86%			17%		33%	29%
2	17%	14%	83%	86%	83%	100%	61%	67%

The most improvement occurred in the area of verbal representations which may be due in part to the fact that the participants were asked to not only solve a significant number of verbal problems but also create a number of verbal scenarios to represent functions, disproportionate to the other representations.

The participants' responses to the prompt "What is a function?" revealed that five of the seven participants possessed a *process* understanding of function while the other two indicated a *dependence* and a *correspondence* understanding of function. Examples of the responses coded as process understanding are as follows:

- *A function is an equation that modifies an input value to produce an output value such that no two outputs are derived from the same input.*
- *A function is an equation or system that produces only one output value for each input value.*

The participant's response that was coded as correspondence stated, "There is only one correspondence from each element in the domain to the range." The participant's response that was coded as dependence stated, "A function is a way of representing an equation where the output depends on the input."

Discussion

Cooney et al. (2010) not only indicate that functions are challenging for students to learn and teachers to teach but go on to say that "students in grades 9-12 need to understand function well if they are to succeed in courses that build on quantitative thinking and relationships" (p.1).

They further state that “the importance of understanding function and the challenge of understanding them well make them essential for teachers of mathematics in grades 9-12 to understand extremely well themselves” (p. 1). In their book, Cooney et al. (2010) identifies five big ideas around which essential understanding of functions is developed. This study focused primarily on one of these big ideas, the notion of multiple representations of function. “Functions can be represented in multiple ways, including algebraic (symbolic), graphical, verbal, and tabular representations. Links among these different representations are important to studying relationships and change” (Cooney, et al., 2010).

The results of this study reveal that this group of secondary mathematics teacher candidates has a limited understanding of the concept of function. Their ability to identify whether or not a representation was a function and provide a satisfactory explanation was not as strong as what might have been hoped for given they are near the end of their teacher preparation program. When presented with a graphical representation, the participants overwhelmingly used the vertical line test to determine if the graph represented a function or not. In line with Clements (2001) findings, the participants in this study also applied the vertical line test to the drawing included in one of the verbal representation problems indicating a lack of sophistication in their understanding of function. Further, in their explanations on the pre/post-assessment, if the participants provided a reason other than the vertical line test they overwhelmingly failed to mention the “single-valuedness” of functions. They frequently indicated that one element from the domain should “produce” or “be aligned with one element from the range” but rarely indicated that this should be a unique relationship or that for each element of the domain, there is exactly one element of the range.

Although this group of teacher candidates has successfully completed a significant number of college-level mathematics courses, their conception and understanding of function is limited and in some cases incorrect. Likewise, the limited nature of their explanations for why a particular representation was or was not a function revealed that they may view mathematics as primarily about computation and rules. These findings align with Wilson’s (1994) suggestion that although it is important for secondary mathematics teacher candidates to consider advanced mathematics topics, it may be more important that they are provided ample opportunity to reflect on their own conceptions and understandings while learning (or re-learning) mathematics they will have to teach.

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CHALLENGES IN THE MATHEMATICS PREPARATION OF ELEMENTARY PRESERVICE TEACHERS

Carole A. Hayata
Texas Woman's University
chayata@twu.edu

Teachers (K-12), knowledgeable in both mathematics content and pedagogy are needed to guide students' learning in both content, and thinking and reasoning skills in this time of extraordinary and accelerating change. The development of mathematical knowledge for teaching (MKT) among 176 undergraduate elementary pre-service teachers during a mathematics methods course and student teaching experience was studied over a nine month period. The results of the study suggested that the participants' did not enter the course with a deep understanding of mathematics concepts and that the absence of mathematical competence hindered their ability to leverage the full opportunity of the course.

Educational policy focused on the improvement of mathematics skills in K-12 students continue to emphasize development of students' conceptual understanding and application of mathematics concepts in addition to the procedural knowledge often associated with rote skills (National Mathematics Advisory Panel, 2008). The emphasis on teaching students a deeper understanding of mathematical concepts and the use of multiple strategies to support answers, has been especially challenging for teachers whose own education in mathematics was most likely centered on rote memorization of facts and rules (Ma, 1999; Stigler & Hiebert, 1999; Schoenfeld, 2008). To address this challenge, teacher preparation programs are providing beginning teachers with opportunities to learn how to use concrete models (e.g. counters, base-ten blocks) to build conceptual understanding and connections between real world problems and the abstract, symbolic notation used in higher mathematics courses. Mathematical knowledge for teaching is the professional knowledge of mathematics necessary for teaching. MKT includes both mathematics content knowledge and pedagogical content knowledge (Ball, Thames, & Phelps, 2008).

Theoretical Framework

The theoretical framework of this study was based upon the (a) research on MKT, and (b) research on teacher behavior. First, the MKT framework, grounded in the work teachers do, is comprised of subject matter knowledge and pedagogical content knowledge. Subject matter knowledge includes (a) common content knowledge (CCK), the mathematics expected of most adults who graduate through the K-12 education system, (b) specialized content knowledge

(SCK), or knowledge of mathematics specifically for teaching, and (c) mathematics on the horizon. For example, knowing how to multiply 35 x 25 using a standard algorithm is CCK expected of most adults. Teachers, on the other hand, also need to know how to detect mathematical errors, determine if methods and solutions different from the ones they are familiar with are valid, ask appropriate questions to probe student thinking and correct misconceptions, and use representations to make connections, all examples of SCK. Pedagogical content knowledge includes (a) knowledge of content and students, (b) knowledge of content and teaching, and (c) knowledge of curriculum. Ma (1999) identified three periods in which teachers' subject matter knowledge develops (a) during their own schooling experiences (students of mathematics, K-12 and college-level), (b) during a teacher preparation program, and (c) while teaching students in the classroom. Although pre-service teachers have limited time and experience in the classroom working directly with students, CCK and SCK can develop without direct student interactions.

Second, teacher behavior and learning is influenced by subject matter knowledge, and attitudes and beliefs towards mathematics, within a social context (Van der Sandt, 2007). When considering teacher preparation, existing knowledge and beliefs are critical factors in determining what and how teachers learn from educational experiences and these beliefs are difficult to change (Kajander, 2010). Furthermore, the level of mathematical competence of pre-service teachers prior to entering the preparation program greatly influences the focus of the program itself. If pre-service teachers do not possess mathematical competence prior to entering the teacher preparation program, then teacher educators must dedicate methods instructional time to school-level mathematics. Teachers in the U.S. are caught in a cycle of low-quality mathematics learning (Ball, 2003; Ma, 1999).

Although studies with pre-service teachers have documented growth in MKT within mathematics content courses designed for teachers (e.g. Mathematics Education of Elementary Teachers (ME.ET), 2009; Welder, 2007), there have been limited studies on the impact of the mathematics methods course on the development of MKT. By design, the objectives of a mathematics methods course focus on both content and pedagogical content knowledge. Therefore, the purpose of this study was to investigate the impact of a one-semester mathematics methods course, and follow-up full time student teaching assignment on the development of MKT. The focus in algebraic reasoning, in particular two underlying components of algebraic reasoning, number sense, and algebraic thinking (Schoenfeld, 2008; Welder & Simonsen, 2011), reflects the importance of mathematics literacy, especially in algebra, for students' future economic independence (Moses & Cobb, 2011).

Methods

The following research questions were developed to guide the study: (RQ1) What is the impact of a mathematics methods course followed by a student teaching assignment on the development of MKT in (a) number sense, and (b) algebraic thinking among undergraduate elementary pre-service teachers, and (RQ2) what are the relationships between changes in MKT in number sense and algebraic thinking and (a) participant demographics, (b) prior knowledge of mathematics, and (c) attitudes and beliefs towards mathematics? Evidence for causality was supported by the collection of longitudinal data from the same panel of participants, data on dependent variables at the beginning of the study, and the use of confirmatory structural equation analysis (Byrne, 2012; Johnson, 2001).

Participants (n = 176) were recruited during fall 2011 semester of their final year of a four-year university teacher preparation program at a large public university in north Texas. Prior to entering the fourth and final year of the preparation program, pre-service teachers are required to take two mathematics content courses (for elementary teachers) in the mathematics department. During fall 2011, participants were enrolled in an elementary mathematics methods course (henceforth called Methods) taught by the College of Education. During spring 2012, participants completed a full-time student teaching assignment (henceforth called Student Teaching) over 15 weeks. The demographics of the study participants (e.g. 79% between 21-25 years in age, 13.1% Hispanic) were representative of the pre-service teachers enrolled in a traditional university teacher preparation program in the United States (USDE, 2011). Although I taught several sections of Methods in prior semesters, I was not teaching any of the Methods sections during the time of the study.

In order to measure each participants' mathematical content knowledge, a computer adaptive test version of the MKT measures developed by the Learning Mathematics for Teaching project (Hill, Schilling, & Ball, 2004) was administered to the participants at five time-points (every six weeks) over nine months. The MKT measures were designed to measure both CCK and SCK, addressed in the Methods course. The multiple time-points were essential to the growth model analysis. The computer adaptive test version allowed for the (a) selection and adjustment of the level of difficulty for each participant, (b) efficient use of time, and (c) reduction in test fatigue. Scores for number sense (elementary number concepts and operations) and algebraic thinking (elementary patterns, functions, and algebra) were reported in standard deviation units based on the expected performance of the average K-8 inservice teacher, mean = 0, SD = 1. The MKT measures have enabled the documentation of growth in mathematical knowledge among elementary teachers participating in professional development and among elementary pre-service teachers following the completion of mathematics content courses designed for teachers. When the MKT measures were used with pre-service teachers, the level of difficulty was adjusted as low as -0.25 since pre-service teachers have had less time in the classroom working with mathematics curriculum (Welder, 2007).

In order to evaluate the influence of participant's prior knowledge of mathematics, and attitudes and beliefs towards mathematics, a background survey and an Attitudes and Beliefs survey (henceforth known as AB survey) was adapted from the *Student Assessment, Parts 1 &* 3 from the ME.ET project. The background survey gathered information on participant demographics and prior knowledge of mathematics (e.g. mathematics courses taken, exam scores, high school GPA). The background survey was administered once at the beginning of the study. Items on the AB survey were grouped into four factors: (a) usefulness of mathematics, (b) multiple ways of doing mathematics, (c) nature of mathematics (rigor and precision), and (d) processes of doing mathematics (enjoyment). The AB survey was administered to all participants at each of the five time-points along with the MKT measures.

To evaluate the performance of the participants on the MKT measures and AB survey, as a function of their progression through Methods and Student Teaching, the data was analyzed using a piecewise growth model. *MPlus* statistical software (Muthén & Muthén, 1998-2010) was used to model the longitudinal developmental trajectories of MKT (number sense and algebraic thinking) and to identify possible influencing factors related to (a) demographics, and (b) prior knowledge of mathematics from the background survey, and (c) attitudes and beliefs towards mathematics from the AB survey. This confirmatory approach founded in structural equation modeling, allowed for the analysis of data for inferential purposes, using longitudinal data over multiple phases (Byrne, 2012; Johnson, 2001). A repeated measures ANOVA was used to analyze changes in the AB factors over the nine months.

Discussion of Findings

RQ1 – The findings from the study suggested that the university teacher preparation program did not positively influence the development of MKT in either (a) number sense or (b) algebraic thinking. In particular, there was no change in participants' overall scores in number sense during Methods and a decrease in Student Teaching (Figure 1). Furthermore, there was a decrease in participants' overall scores in algebraic thinking during Methods and no change during Student Teaching (Figure 2).

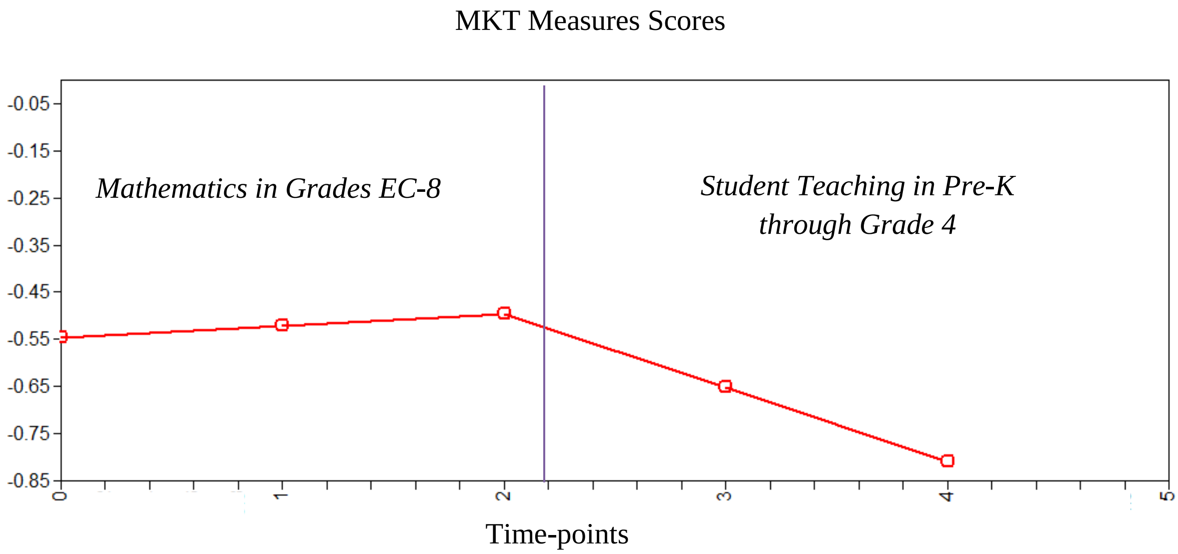

Figure 1. Piecewise Growth Model with Estimated Means – Number Sense. The mean initial MKT measures score in number sense was -0.55 (S.E. 0.05). Although there were no statistically significant changes during Methods (time-points 1-3) (slope = 0.03, S.E. = 0.03, p = .392), there was a statistically significant decrease during Student Teaching (time-points 4-5) (slope = -0.16, S.E. = 0.03, p < 0.001).

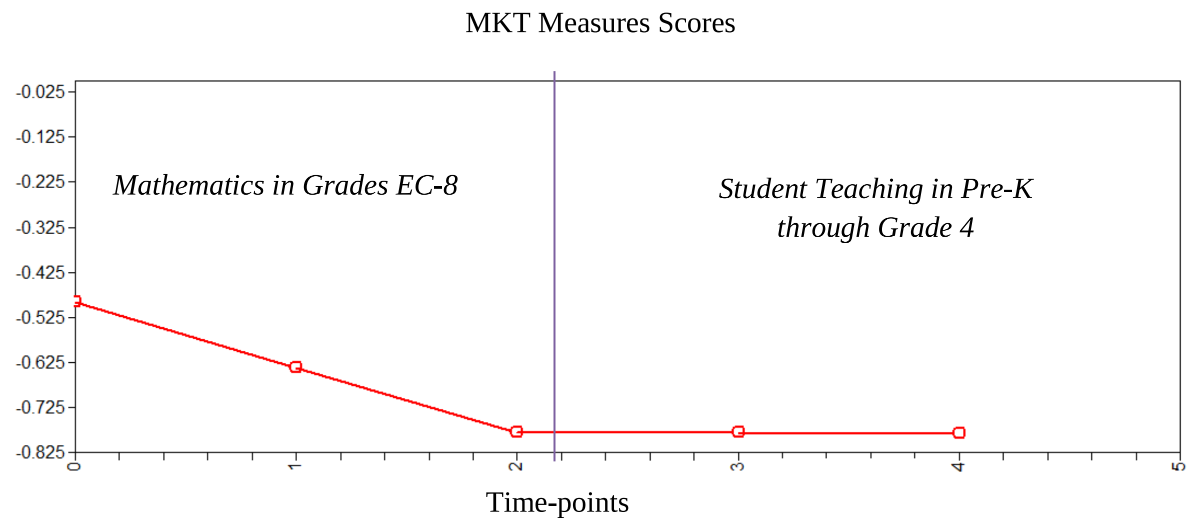

Figure 2. Piecewise Growth Model with Estimated Means – Algebraic Thinking. The mean initial MKT measures score in algebraic thinking was -0.47 (S.E. 0.06). There was a statistically significant decrease during Methods (time-points 1-3) (slope = -0.16, S.E. = 0.03, p < 0.001), though there was no statistically significant change during Student Teaching (time-points 4-5) (slope = -0.02, S.E. = 0.04, p = 0.653).

RQ2 – The statistically significant covariates (i.e. demographics, prior knowledge of mathematics, and AB factors) identified for the initial MKT scores were as follows: (a) for number sense, *Hispanic*, and the number of *Advanced HS Math courses* taken, and (b) for algebraic thinking, the AB factor *processes of mathematics* (Table 1). The initial MKT scores are indicative of the participants' mathematical knowledge developed over their K-12 schooling and college level mathematics courses. In number sense, participants who identified themselves as Hispanic scored lower on the MKT measures while those who took more advanced high school mathematics courses and believed mathematics was enjoyable and creative scored higher at the beginning of the study. While in algebraic thinking, participants who viewed mathematics as enjoyable and creative scored higher on the MKT measures at the beginning of the study.

Table 1.

Influence of Participants' Background & Attitude and Beliefs

	Number Sense		Algebraic Thinking	
Intercept (time-point 1)	Estimate (*SE*)	95% C.I.	Estimate (*SE*)	95% C.I.
Hispanic	-0.36* (.13)	[-0.61, -0.10]	-0.35 (.16)	[-0.66, -0.04]
Advanced HS Mathematics	0.16* (.05)	[0.06, 0.27]	0.11 (0.07)	[-0.01, 0.25]
AB Factor: Processes of doing mathematics	0.20 (0.09)	[0.03, 0.37]	0.33* (0.11)	[0.12, 0.54]

Note: * statistically significant after Bonferroni correction at p <.05

The analysis of the four AB factors from time-points 1 to 5 using repeated measures ANOVA suggested that over the nine month study, participants believed more strongly that there was a single way to solve problems ($p < 0.001$) despite the learning experiences provided by the Methods instructors to demonstrate multiple ways of solving problems. In addition, participants found mathematics more enjoyable ($p < 0.001$) over nine months. The regression in participant mathematical reasoning and the greater focus on finding the right answer may have limited participants' ability to think out of the box and to think about higher mathematics effectively.

Overall, the participants' lack of growth may have been due to incorrect assumptions about incoming pre-service teachers core mathematical skills. Participants had a weaker mathematical background entering Methods relative to other studies of pre-service teachers using the MKT measures. These previous studies either used the measures as published (mean = 0), or created forms using MKT items with mean = -0.25 (ME.ET, 2009; Welder, 2007). The initial MKT scores for the study participants were: (a) mean = -0.55 for number sense, and (b) mean =-0.47 algebraic thinking. The Methods course was designed based on prerequisites that may have been inconsistent with the actual pre-service teachers' mathematics aptitude. Thus, participants may have been incapable of leveraging the full opportunity of Methods.

The results of the study were discussed with a panel of three university Methods instructors (all full-time faculty). The instructors believed that the attitudes and beliefs of the participants towards mathematics may have contributed to their low performance. Elementary pre-service teachers are characteristically fearful of mathematics due to lack of success as a student of mathematics themselves. In addition, the high cognitive demand of the MKT measures combined with a low level of mathematics understanding may have resulted in frequent guessing. Participants may have also lacked motivation when completing the measures as they were not tied to their course grade. Suggestions for further study included: (a) administer the MKT measures to pre-service teachers as they enter the first mathematics content course, then increase the time between time-points ending with Student Teaching, (b) provide a greater incentive to motivate participants, and (c) conduct personal interviews with participants to gain further insight as to how participants perceived the measures and how they selected their answers.

Although the participants' demographics were typical of students enrolled in a traditional elementary teacher preparation program, the initial low-level of mathematics knowledge and deep rooted belief in a single way of solving mathematics problems may have limited the impact of the Methods course. In order to successfully match learning strategies and activities with students, situations, and opportunities, teachers must have a deep understanding of mathematics content, a large repertoire of pedagogical strategies, and the ability to make decisions about which tool will be the most effective in a given situation.

In practice, the three periods of a teachers' development of mathematics knowledge is greatly influenced by the pre-service teachers' own schooling. The relatively short period of time spent in the teacher preparation program may only be able to begin reshaping their prior experiences. Based on the findings of this study, recommendations include (a) the re-evaluation of minimum teacher preparation program entry requirements for mathematics content knowledge, and (b) review the current Methods curriculum, and (c) for new teachers already entering the field, participation in continued professional development focused on both mathematics content knowledge and reform-based pedagogy in order to strengthen teachers conceptual knowledge of mathematics and to continue to peel away deep-rooted beliefs towards mathematics.

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THE INTERVIEW PROJECT

Angel Rowe Abney
Georgia College & State University
angel.abney@gcsu.edu

Doris Santarone
Georgia College & State University
doris.santarone@gcsu.edu

In a mathematics content course for preservice early childhood teachers (PSTs), an interview project is used to analyze whether the PSTs can learn to listen to, analyze a child's mathematics, and inform their mathematics and teaching. Our study suggests that this project is an effective tool for changing PSTs views of the discipline of mathematics and what it means to teach mathematics.

In the teaching principle, the National Council of Teachers of Mathematics (2000) called for teachers to have not only a deep understanding of the content that they teach, but also knowledge of their students. To make this type of reform in mathematics education possible, Thompson and Thompson (1994) contended that classroom discourse and communication are essential elements. They suggested that teachers "must be sensitive to children's thinking during instruction and shape their instructional actions accordingly—to ensure that children hear what they intend them to hear" (p. 279). If the current practice of assessment and instruction is to change, how might this change occur? Where does such reform begin? We believe it should begin with preservice teacher (PST) education.

As instructors of the first mathematics content course in a sequence of four courses for early childhood majors, we want the PSTs to develop an intuitive number sense. We also believe that they need to work toward communicating using precise mathematical terminology to construct and justify arguments. A major part of the work of teachers is to interpret their students' solutions and determine the reasonableness of answers and evaluate the efficiency of methods.

While the early childhood program is field-intensive, unlike many elementary education programs, the PSTs in our program never take a mathematics specific methods course. Because we feel passionate about integrating pedagogical experiences into their content courses, we decided to create an experience where PSTs were required to listen to and learn from children. Ultimately we want PSTs to allow what they learn from children to influence how they think about their own mathematics and allow it to inform their teaching. In this sense they will be assessing a child's developmental level with respect to whole numbers. For this project the PSTs are required to describe a child's mathematics, analyze the child's mathematics using the framework from the course, apply their analysis to inform their instructional decisions (if they were to work with this child again), and to discuss any on-the-spot instructional decisions they made while working with the child.

Field experiences are repeatedly identified as the most significant part of teacher preparation programs (McIntyre, Byrd, & Fox, 1996; Mewborn, 2000). However, more field experiences do not always lead to productive growth for PSTs (McIntyre, et. al., 1996). Experiences in the field can simply be used to socialize PSTs into traditional ways of teaching mathematics, and PSTs rarely have opportunities to reflect critically about these experiences (Mewborn, 2000). Typical field experiences take place in a classroom full of students. In this setting, the focus tends to be on management concerns, and PSTs do not get to focus on mathematical content and making sense of children's mathematics (Mewborn, 1999). Our hope is that having them work with and reflect on one or two children's mathematics, they will be more ready to focus on what students know and how they learn while they are engaging in their field experiences each semester.

Literature Review

As the PSTs are engaged in an interview with one or two students, we are aware that this is one of their first experiences in listening to and responding to children in mathematically productive ways. "Clearly, the act of unpacking learners' mathematics requires listening to students" (D'Ambrosio, 2004, p. 139). Davis (1996) suggests that while you can't observe listening occur, you can infer how a teacher is listening through how they respond to students. You can also infer how a teacher is listening by what they are listening for and what they choose to ignore. Questioning is one way that teachers respond to students. Questions are instructional decisions that can often occur on the spot or some may be planned in advanced. Questions can be categorized in three different ways: probing, prodding, and prompting. Probing involves questions to determine what or how a student is thinking, prodding questions are intended to keep a student acting mathematically, and prompting questions attempt to elicit a specific response or strategy to a task (Abney, 2007).

We also encourage the PSTs to construct a model of their child's mathematics, where they can both describe and analyze the child's mathematics, we have found it helpful to provide them with a conceptual framework of children's whole number development. We discuss the Cognitively Guided Instruction (CGI) framework for classifying story problems, counting strategies, or other strategies. The CGI researchers identified several problem types through their interactions with children. We expose our PSTs to four basic structures for story problems involving addition and subtraction: join, separate, part-part-whole, and compare. The join and separate problems involve an action and can have a result unknown, a change unknown or an initial unknown. The part-part-whole and compare problems have no action, which tends to make those more difficult for children to solve (Carpenter et. al, 1999).

There are additional frameworks from the work of Steffe and von Glasersfeld (1983; 1988) to help teachers to consider how children's mathematical thinking develops from direct modeling strategies to more abstract and sophisticated strategies. These frameworks help teachers think about how to use children's current ways of operating to inform their instructional decisions. Children often begin solving story problems by directly modeling the story with counters or using *counting all* strategies (Olive, 2001). It is not until children are able to see a group of objects as a unit that they are able to *count on* to solve story problems. Children at this level are said to be numerical and are counters of abstract unit items. Since counting is no longer rote for them, they are said to have constructed their initial number sequence (INS). At the next numerical stage, which can be characterized as INS Plus, children are able to use the *counting down* strategy more effectively and they have now determined that it is more efficient to solve an addition problem by *counting on from the largest* number in the problem rather than the first. Thus, these children have constructed the commutative property of addition. Children who are at the next level can solve all types of problems without the use of counting. They are able to take numbers apart and put them back together in more convenient ways. They are said to be Strategic Additive Reasoners (SAR) (Steffe, et. al, 1983). The CGI researchers call these strategies using number facts.

Methods

We wanted to systematically study what PSTs get out of the interview project within the context of a subject matter preparation course. In particular, we wanted to know if the project design meets the goals we set. For instance, did the PSTs learn to listen to children to inform their mathematics and their instructional decisions? Were they able to identify counting schemes or strategic additive strategies that the child used when solving the CGI story problems? Were they able to use their own mathematics to recognize the mathematical validity of a child's method?

There were 26 participants in our study, all of whom were taking a Numbers and Operations Course designed specifically for Prospective Early Childhood PSTs. For all of the PSTs, this was the first time that they worked with a child to learn to listen to and be responsive to the child's mathematics. This project involved PSTs working in pairs to interview an elementary-age child to allow them the opportunity to see how capable children are of solving problems. We analyzed the work of thirteen pairs by coding instances of description, analysis, and instructional decisions. Sources of data came from PSTs' written reports along with their presentations of the interview, and their peers' responses to their presentation. We were particularly looking for PSTs who seemed to be striving to understand their own teaching, their child's mathematics, and the way in which it can inform their practice.

Findings

Description and Analysis

All PSTs were able to describe the children's mathematics using the language that they learned in class. Their descriptions were informed by frameworks including CGI and Steffe et. al. They used language such as direct modeling, counting on, counting all, INS or SAR. In their written reports the PSTs tended to simultaneously describe the child's mathematics and analyze their descriptions. One PST wrote:

> …she was definitely numerical. Alex was able to count with the counters and had no trouble counting on. She also used strategic reasoning for many of the problems and was able to explain to me how she worked it out….When I asked her how to solve six plus seven, she said she knew this "because six plus six is twelve and one more for seven." This is a perfect example of "near doubles". She used the basis of a double she knew and added on one. When I asked her four plus nine, she knew this was thirteen because "ten plus four is fourteen and take away one". She used the base of ten and added on from this.

This PST was able to precisely describe what the child did to solve addition problems; she was able to name these strategies, such as counting on and strategic reasoning. She was also able to correctly identify that the child's mathematical actions indicated that the child was numerical. During the class presentation, the PST was able to write a series of equations and name the mathematical properties in order to analyze the validity of the child's strategic reasoning. This is the series of equations that she wrote:

6+7	4+9
=6+(6+1) Substitution	=4+(10-1) Substitution
=(6+6)+1 Associative Property	=(4+10)-1 Associative Property
=12+1	=14-1
=13	=13

Instructional Decisions

Even though this was the first time the PSTs had worked with children in this capacity, it was evident that many instructional decisions were made, both planned and spontaneous. All reports included at least one spontaneous instructional decision. One example was based on what a pair of PSTs observed their child do to confirm their belief that the child was at the INS+ level of whole number development. This pair of PSTs wrote the following in their report:

> Her approach proved to me that she can use the commutative property. She began with the larger number although it appeared after the smaller number in the problem. To reinforce this I asked her a similar question (I replaced the numbers with 6 and 9 in that order) and she still chose to start with the 9. I asked her why she chose to start with the 9 instead of the 6 since it was first in the problem and she told me it was easier to start with 9 because it was the bigger number. This supported my belief that she understood the commutative property and was at least INS+.

This excerpt shows the PSTs recognition of the mathematical properties that often were the focus of the content course on Numbers and Operations. It was evident that they were able to make their content knowledge usable in their work with children. They were also able to use the framework from class to analyze the child's mathematics. However, in order for this analysis to occur they had to use questions to probe the child's mathematics. It was clear that these PSTs were intentional with their choice of ordering the numbers in the problem. It was clear that these PSTs were listening for a particular strategy, specifically counting on from largest. They had a hypothesis, and used a specific task to test that hypothesis. We believe that this is an important part of the research process. Besides probing questions, which are information seeking, we also saw evidence that the PSTs used prompting questions to elicit a particular response.

On the project description we asked the PSTs to respond to the question, "if you could continue to work with this child, what concepts or kinds of problems do you think would be productive work for her or him?" 12 out of 13 pairs were able to thoughtfully respond to this question. However, 8 pairs weren't able to pinpoint any specific concept or relate their instructional decision back to the framework. In this excerpt it seems clear that these PSTs are searching for what might be on the cusp of what is possible for the child.

> If we could continue working with this child [we would give him] problems or concepts involving subtraction with greater values and more division problems so that he comprehends he is actually doing division. Other problems that would be applicable to this student are those involving remainders and fractions. This is where his ZPD is located. We think this because he can do both multiplication and division sufficiently and accurately, but we are curious as to whether he fully comprehends these concepts. If we could see him work with numbers in fractions and as a part of a whole we might get a better feel for his future potential and understanding of the material.

Four pairs of PSTs were able to specifically address the levels in the framework and suggest directions for the child's mathematics related to particular types of word problems.

> If I were to continue to work with Allyson, I would probably encourage her to explore more strategic ways to solve problems. She depended on the counters for much of the interview, which suggests she is not entirely comfortable with solving word problems in her head or without some sort of physical visual. Perhaps with more practice and more strategic reasoning, Allyson could move from the INS+ level to SAR.

While this pair of PSTs was able to discuss pushing the child from the INS+ Level to the SAR Level by getting the child to become less reliant on physical materials, they were not able to give specific problems or tasks that would encourage this next level of reasoning. One such example might be using a cup or some cover so that some objects are left unseen.

Discussion

The PSTs in our interview project were able to describe and analyze the children's mathematics using the framework from class. They were also able to make some instructional decisions. However, because any future instructional decisions were hypothetical, they lacked focus and were mostly explorational. This leads us to ask the question, would it be worth the time and additional efforts from our students' perspective to have them conduct at least a second interview with the same child. We feel that this would give them the opportunity to go beyond the interview and force them to make the instructional decisions.

We strongly believe that this interview project has a great impact on the PSTs and will change the way they previously felt about teaching mathematics. Some of the typical reflections from the PSTs about the overall project highlighted their connection of theory to practice, seeing different ways to do mathematics, and making instructional decisions based on children's mathematics. One pair of PSTs noted:

> We learned a lot about how to apply different concepts like counting from largest, different kinds of word problems, how to figure out a child's strategy, and so on. We not only learned a lot from having to analyze everything Kylie was doing, but Kylie actually ended up teaching us a lot as well! We learned new strategies and realized how much children can teach you about how they think. They can analyze problems so much differently than adults, and sometimes their way seems to work better. In conclusion, we not only cannot wait to begin teaching students, but in turn, learn from our students.

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AN INNOVATIVE APPROACH FOR SUPPORTING AT-RISK STUDENTS IN ALGEBRA I

Judith Olson
University of Hawaiʻi
jkolson@hawaii.edu

Melfried Olson
University of Hawaiʻi
melfried@hawaii.edu

Hannah Slovin
University of Hawaiʻi
hslovin@hawaii.edu

Linda Venenciano
University of Hawaiʻi
lhirashi@hawaii.edu

Fay Zenigami
University of Hawaiʻi
zenigami@hawaii.edu

The Curriculum Research & Development Group has developed A Modeling Approach to Algebra, a curriculum created to support ninth-grade students' effort to learn Algebra I. Funded by a contract with the Hawaiʻi State Department of Education, materials were developed to support struggling learners by emphasizing modeling mathematical content and practice as described in the Common Core Curriculum Standards for Mathematics. In this paper we discuss the curriculum research and development from a design research perspective.

To successfully complete the mathematics requirements created by adopting the Common Core State Standards in Mathematics (CCSSM) (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) raises the bar for students. This is of particular concern for Algebra I, the traditional entry point to high school mathematics. Such is the case in Hawai'i where pre-algebra is no longer a high school course. Partially as a result of these policies, approximately one in three students do not succeed in high school Algebra I (Gottlieb, personal communication, Spring 2011). To address the Algebra I failure rate, a course, *Modeling our World* (MOW), focusing on modeling and opportunities to learn mathematics in a more investigative manner was established. Although designed for struggling learners, MOW is not remedial and is intended to be taken concurrently with Algebra I.

The Curriculum Research & Development Group (CRDG) at the University of Hawai'i was contracted to design and develop the curriculum materials for the MOW course. The CCSSM together with CRDG's previous curriculum research and development projects, e.g. *Algebra I: A Process Approach* (Rachlin, Matsumoto, Wada, & Dougherty, 2001), *Reshaping Mathematics for Understanding* (Slovin, Venenciano, Ishihara, & Beppu, 2003), provided a research base from which to begin the development for MOW. The modeling standards embedded in the CCSSM were established as the framework around which to build the materials. This paper describes the research and development process for developing the curriculum materials, *A Modeling Approach to Algebra* (AMAA) (Curriculum Research & Development Group, 2012).

Background

Since the term mathematical modeling has various meanings in curricular discussions and implementations, previous research and resources were examined to begin the research and development process (Galbraith & Stillman, 2006; Indiana Mathematics Initiative, 2012; and Lesh & Zawojewski, 2007). AMAA follows Lesh and Zawojewski's suggestions that students begin their learning experiences by developing conceptual models for making sense of real-life situations and then create, revise, or adapt a mathematical way of thinking by using modeling for problem solving. In this way, students simultaneously gain an increased understanding of both the problem situation and their mathematization of the problem.

Initially, nearly 60 HIDOE mathematics department chairpersons and other school and curriculum leaders organized in focus groups responded to questions addressing three areas: student preparation for algebra, students' use of modeling in mathematics, and an effective course of study. Major emerging themes were bridging concrete and abstract representations, language and communication issues, and the need to build students' affective domain.

Curriculum Framework

Materials in AMAA are designed around the premise that learning algebra requires more than memorizing formulas and finding answers. The development of the materials followed five tenets foundational within all CRDG mathematics curriculum projects:

> (a) problem solving is the method of instruction to introduce new topics or concepts; (b) communication through reading, speaking, writing, critical listening, and representing mathematics in multiple ways helps students clarify, validate, or refute ideas; (c) development of understanding from a conceptual level to a skill level occurs over time; (d) new learning experiences are built upon previously developed understandings with common threads running throughout; and (e) challenging but accessible problems having multiple solutions at varying levels of complexity (open-ended) allow children of diverse abilities to respond (Slovin, Rao, Zenigami & Black, 2012, p. 4).

The lessons emphasize the use of models, promote the investigation of open-ended problem solving tasks, and provide appropriate pacing for students to develop concepts, generalizations, and skills. In addition, there is heavy emphasis on the CCSSM eight Standards for Mathematical

Practice throughout the investigations and communications. Students are asked to model, represent, graph, write about, and discuss their strategies for investigating and solving problems as they begin to internalize algebraic ideas and develop an understanding of algebraic techniques.

Methodology

Design research is highly interventionist and requires researchers to work closely with teachers while collecting extensive feedback and data for re-design and revision (Cobb, Confrey, Disessa, Lehrer, & Schauble, 2003). Circumstances of working with the HIDOE necessitated an adaptation of the design phase. Our adaptation of this approach for the research and development of AMAA began with an initial design and development followed by implementation of materials in a small set of classrooms. During this process there was a review of materials and discussions between researchers and teachers implementing the materials. From these results the materials are being revised and will be re-implemented in other settings.

Aligned with the CCSSM

The content and practice of modeling provide coherence for the lessons. High school standards specific to modeling and appropriate for the Algebra I course of study were selected as the basis for AMAA. The resultant curriculum is a mixture of problems and investigations situated in real and practical settings where students experience mathematics in accordance with the modeling cycle diagram in the CCSSM (2010, p. 72). Extended explorations and problems from pure mathematics are also included.

The AMAA content is organized according to the five critical areas identified for Traditional Pathway: High School Algebra I, Unit 1 Relationships Between Quantities and Reasoning with Equations, Unit 2 Linear and Exponential Relationships, Unit 3 Descriptive Statistics, Unit 4 Expressions and Equations, and Unit 5 Quadratic Functions and Modeling. A preliminary unit, Unit 0 Getting Started, introduces students to problem solving investigations and processes used in the course. Because students for whom this course was intended often do not have experience conducting mathematical investigations, Unit 0 problems highlight modeling, specifically, the modeling cycle suggested by the CCSSM (2010, p. 72–73). Lessons provide opportunities for the class to establish norms for an environment critical for productive classroom discourse. Unit 0 also initiates the focus on standards for mathematical practice that students will be expected to embrace with greater proficiency as they progress through the materials.

Format of AMAA lessons

The investigations introduce and develop concepts through carefully constructed problems. These investigations give students the opportunity to use aspects of modeling to interpret problematic situations; understand the goals of a problem; conjecture, represent, test, and revise various approaches to solving the problem; and report on results. Students are encouraged to offer alternate solutions and solution methods, question others' methods and results, and reflect on their own understanding. Major student projects are included in Units 3, 4, and 5 and are designed to be more open-ended to encourage students to employ problem solving skills while investigating complex problems.

Digital files

Technology that provides students the opportunity to interact with dynamic representations of concepts for classroom instruction is integrated throughout the curriculum. The use of technology focuses on using graphical representations for data, encourages conjecturing and validation, and emphasizes relationships between quantities. Prepared documents in TI-Nspire Teacher software include lessons with specific TI-Nspire™ documents (i.e., a .tns document) with suggestions in the teacher materials for how to use them during instruction. These documents are intended to develop the beginning concepts or enhance and extend algebraic ideas of the lessons.

In addition to print materials, student pages, teacher notes, and annotated student pages are also formatted using the TI-Nspire PublishView™ feature of TI-Nspire Teacher software for teachers' instructional purposes. Documents are linked so teachers can use TI-Nspire Teacher software to present a problem to students; link to Teacher Notes or Annotated Students Pages for assistance during instruction; link to an interactive TI-Nspire document for whole class discussion; or, if available, send to students' TI-Nspire handhelds. Occasionally, links to websites are provided to introduce a problem or for background information.

The development of the teacher materials reflects our work in numerous professional development projects (Olson, Zenigami, Slovin, & Olson, 2011) and feedback from teachers designated to implement the AMAA materials. Teacher notes are designed and developed as educative materials (Davis & Krajcik, 2005) and include annotated student pages for each lesson. As educative materials, the teacher notes clarify the mathematical ideas students are expected to learn, and when appropriate include explanations of the mathematics beyond what the students are expected to pursue. Teacher notes are intended to help with lesson planning by providing a summary of the content and objectives for the investigation, highlighting opportunities to model with mathematics, and anticipating student thinking and possible responses—including common misunderstandings. The notes list materials needed and describe ways that technology can enhance student learning as well as provide an alternative approach to understanding the relationships within the lesson.

The annotated student pages expand the material from the student book with notes for managing the investigation and suggest questions to prompt discussion. Questions are intended to indicate topics and ideas important to the investigation. As students become familiar with the instructional approach, they are expected to raise these issues themselves or pose the questions spontaneously to extend a problem or probe its mathematical content.

The investigative, problem-based approach changes the roles of teachers and students. The suggested pedagogy is student-centered, with students and teacher sharing ideas in the classroom mathematical community. Students should be explaining their thinking, questioning their own and others' ideas, and analyzing suggested strategies. The teacher should orchestrate the discussion with thought-provoking questions, select examples of student work to be shared when doing so furthers the learning opportunity, and provide suggestions for techniques of mathematical inquiry and discussion when students need guidance.

Implementation of materials

Feedback on teacher implementation of the initial set of materials is used to inform the development work and to determine if what is taking place in the classroom matches the intent of the course. In response to an announcement about the initiation of the MOW course accompanied by course materials and teacher training, 17 teachers from high schools throughout the state participated in professional development in Summer 2012. Seven of those teachers and four additional teachers not trained during the summer are implementing AMAA in MOW courses. Support for these teachers also includes four follow-up sessions during 2012–2013 to learn more about AMAA curriculum materials and approaches for teaching lessons, modeling and the CCSSM, student expectations, and technology integration. During follow-up sessions, teachers share with and learn from other teachers, ask questions and pose teaching problems, and provide valuable feedback to CRDG researchers. Data is collected on such matters as the appropriateness of lessons by discussing lessons that worked well and lessons that were problematic or in which engaging students was difficult; work of students; the suggested length of time for each lesson; appropriateness of physical materials; critiques of the use of technology; and suggestions for changes in the materials. Upon the completion of each unit, teachers provide feedback to specific items via Google forms.

Data Sources for Curriculum Revision

The initial implementation of AMAA has provided several opportunities for the research and development team to collect data for curriculum revision. Although MOW was intended to be co-requisite to Algebra I, most schools have allowed students to take just the MOW course. Due to this situation, the information received regarding the appropriateness of the materials has not been as useful as desired. Data sources used to develop insights for curriculum revisions include classroom observations, professional development sessions, teacher reports, and student work.

Classroom observations

CRDG researchers are conducting classroom visits with the expectation to complete one per quarter per pilot teacher. An observation form is used as a guide to note how students are engaging in lesson activities, ways teachers are presenting MOW investigations, the nature of classroom discourse, how teachers are using MOW resources to deliver lessons, and what technology is being used and in what manner. The person conducting the visit then completes an online version of the form.

Feedback during follow-up sessions

At the beginning it was difficult for teachers to use the problem solving approach and implement the modeling cycle. However, over time, teachers are sharing ways that students are becoming more accustomed to this. Teachers see success in the quantity and quality of student work and in student communication.

Teacher reports

Some teachers have reported steady progress on their students' work while others continue to report students are slow to embrace the content and style of the course. One special education teacher, "Laile", has had success in using the questions and format of the basic modeling cycle. She posted these in the classroom and regularly refers students to them. She created and displays a Problem, Formulate, Compute, Interpret, Validate, and Report poster with descriptions of each along with a rubric, and reminds students to use those for their reports. Her poster reminds students to organize their reports "so the students can get into the habit of looking at the problem and devising their own questions in their thought process."

Student work

We identified lessons for teachers to collect and submit students' work. Teachers were asked to select at least one student each at high, middle, and low levels of achievement. For example, while most students emphasize the answer and the steps taken to generate their answer without describing why the process was valid, Laile is primarily focusing on getting students to communicate what they are doing. She shared students' work and how they were writing reports and compared several examples of earlier work, where students wrote a few sentences, to their current work, which filled a page or more and was organized neatly.

Impact on Revisions

From the data collected through the professional development, observations, teacher reports, and examples of student work there are several changes being planned.

Use of the modeling cycle

Other teachers have embraced the use of the modeling cycle suggested by Laile for students to organize their work and write reports with similar success. The use of the modeling cycle will be more prominent in the revised materials.

More explicit guidance for teaching lessons

Teachers desire more direct information on conducting a lesson than has been provided. This is especially crucial due to the demands made on both the teacher and students related to the problem solving approach used in AMAA. These changes will be reflected in both the Student Pages and in the Annotated Student Pages.

More explicit discussion of the mathematics needed for teaching

Teachers have requested more in-depth explanations of the bigger picture of the lesson content in the Teacher Notes. They desire more connections in the unit overviews between the content within the unit and the content across units. They also requested more answers with explanations.

More explicit guidance in how to use the digital components

Teachers report not using the digital components very much, either because they have not taken time to prepare for its use or that they are not sure how to use it. Now that they are seeing how the PublishView software and the .tns documents can be used, teachers see the relevance of their inclusion and desire more professional development on their use.

Conclusions and Implications

Throughout the process of curriculum research and development using a modified design research model, the project has been able to create curriculum materials based on the modeling cycle. The initial implementation suggests that this approach appears to serve struggling students and their teachers. It provides struggling students problems that are accessible and include structures that allow for appropriate scaffolding for the students. The AMAA problems allow teachers to include substantial work for students on the standards for mathematical practice.

The process of creating AMAA has generated useful insights for the design research process, and allowed the creation and validation of material appropriate for at-risk students. Our revised materials will be useful for other districts facing similar challenges that AMAA addresses.

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FOSTERING PRE-SERVICE TEACHERS' MATHEMATICAL EMPOWERMENT: EXAMINING MATHEMATICAL BELIEFS IN A MATHEMATICS CONTENT COURSE

Mary Harper
East Central University
mharper@ecok.edu

Stacy Reeder
University of Oklahoma
reeder@ou.edu

This study explored the influence of curriculum and experiences in a mathematics content course had on pre-service teachers' mathematical empowerment as reflected in their beliefs about mathematics and mathematics teaching and learning. The results indicate that the pre-service teachers increased their feeling of mathematical power and their beliefs about mathematics teaching and learning were impacted by a college-level mathematics course taught in a non-traditional manner.

Empowerment denotes "the gaining of power in particular domains of activity by individuals or groups and the processes of giving power to them, or processes that foster and facilitate their taking of power" (Ernest, 2002). The National Council of Teachers of Mathematics (1989) introduced the idea of empowerment related to mathematics teaching and learning to a much broader audience of mathematics educators with the publication of the Curriculum and Evaluation Standards for School Mathematics. NCTM stated the following:

> Mathematical power denotes an individual's capabilities necessary to explore, conjecture, and reason logically as well as the ability to use a variety of mathematical methods effectively to solve non-routine problems. This notion is based on the fact that mathematics is more than a collection of concepts and skills to be mastered. It includes methods of investigating and reasoning, means of communication, and notions of context. In addition, for each individual it involves the development of personal self-confidence (NCTM, 1989, p. 5).

Developing the mathematical power of students is a noteworthy goal but unfortunately it is predicated on the notion that teachers of mathematics themselves are mathematically empowered; that their beliefs about the nature of mathematics, how mathematics should be taught, and their own mathematics abilities allow them to be confident and flexible. They must be able to solve non-routine problems and hold a view of mathematics that it is more than a collection of concepts and sills to be mastered. In this way, there is an inextricable connection between one's beliefs about mathematics teaching and learning and one's mathematical empowerment.

The research presented in this paper encompassed challenging pre-service teachers' beliefs about mathematics and mathematics teaching and learning while asking them to reflect upon their beliefs. The study took place with students in a university level mathematics content course focused on number theory, sets, and functions that integrated content and pedagogy as Cooney (1999) suggested. The structure of the course was consistent with reform teaching practices as opposed to a traditional university mathematics classroom structure. The following research question guided the study:

> What influence, if any, do pre-service teachers believe the curriculum and experiences in this mathematics course have on their mathematical empowerment as reflected in their beliefs about mathematics and mathematics teaching and learning?

Related Literature

The notion of empowerment, in mathematics education literature, is often used to mean the same as autonomy or efficacy. In order to avoid confusion, the word empowerment will be used solely in place of the other two throughout this paper.

Mathematical, Social, and Epistemological Empowerment

Mathematical empowerment involves gaining power over the domain of school mathematics which entails using and applying the language, practices, and skills of mathematics; likewise, it has cognitive and semiotic perspectives which are complementary (Ernest, 2002). The cognitive psychological perspective of mathematical empowerment involves the procurement of concepts, skills, facts, and general problem solving strategies whereas the semiotic perspective demands the development of power over the 'texts' of mathematics. These powers over the 'texts' of mathematics include the abilities to read and make sense of mathematical tasks, transform text into smaller tasks, pose problems and write questions, and make sense of text in computational form (Ernest, 2002).

Social empowerment encompasses the use of mathematics to increase a person's life chances and critical participation in work, study, and society (Ernest, 2002). In a utilitarian way, throughout history success in mathematics (often judged by performance on examinations) serves as a 'gatekeeper' or 'critical filter' controlling access into further education as well as occupations with greater pay (Ernest, 2002; Lemann, 1999; Oakes, 1985; Oakes, Ormseth, Bell, & Camp, 1990; Stanic, 1986; Standards, 1989). Moreover, researchers have long noted the perceived inequity in mathematics education for women and other minorities (Fennema &

Sherman, 1977; Oakes, 1985; Oakes, Ormseth, Bell, & Camp, 1990; Sells, 1976; Walkerdine, 1997).

Epistemological empowerment concerns both one's confidence in the use of mathematics and a "personal sense of power over the creation and validation of knowledge" (Ernest, 2002, p. 8). It is in this category that the professional empowerment (or pedagogical empowerment) of the mathematics teacher falls. For many teachers and students, past experiences supports and sustains their belief that knowledge is created, legitimized, and exists outside of themselves. It is with this conception of empowering the learner that teacher mathematical empowerment can be seen as equally vital.

Pedagogical Empowerment

Pedagogical empowerment (or professional empowerment) refers to teachers developing into autonomous and reflective participants in education. Empowered teachers contain the confidence to critically assess and construct mathematics teaching and learning experiences with and for their students (Ernest, 2002). Szydlik, Szydlik, and Benson (2003) found that the culture and socio-mathematical norms of the classroom affected a change in pre-service teachers' mathematical beliefs as well as served to further their autonomy. Socio-mathematical norms established in the classroom are distinct from social norms in that they are unique to mathematics classrooms (Yackel & Cobb, 1996). For example, adequate justification is a social norm in many subject areas but what constitutes as *relevant* and *elegant* for proof of a claim remains exclusive for mathematics. Additionally,

> what becomes mathematically normative in a classroom is constrained by the current goals, beliefs, suppositions, and assumptions of the classroom participants. At the same time these goals and largely implicit understandings are themselves influenced by what is legitimized as acceptable mathematical activity. (Yackel & Cobb, 1996, p. 460)

The socio-mathematical norms established in the classroom studied by Szydlik et al. (2003) were shown to affect their participants' autonomy. These participants indicated that they were "now aware that mathematics is a human creation and they can be a part of making mathematics themselves" (p. 272) in a culture that views mathematics as making sense. Additionally, Anderson and Piazza (1996) found that a classroom practice that eliminated lecture as the main form of instruction together with the use of physical models (manipulatives, pictures, diagrams) served to reduce students' anxiety about learning and teaching mathematics and increase students' confidence. Because beliefs are socially and contextually constructed, many preservice teachers' views of teaching mathematics are consistent with the ways in which they experienced mathematics learning (Ball, 1990; Cooney, 1999). For many preservice teachers, the beliefs they bring with them are created from an "apprenticeship of observation" (Anderson & Piazza, 1996) during their many years of schooling (Ball, 1988; Ball, 1996; Calderhead & Robson, 1991; Philipp, 2000). Consequently, it is possible that the culture of the classroom can contribute to the empowerment of pre-service teachers mathematically and pedagogically. This study sought to explore the relationship between beliefs about mathematics and mathematics pedagogy and the impact on preservice teacher empowerment as told from their perspective. This study sought to intentionally explain the participants' experiences from their perspective as much as possible in the vein of their own words.

Methodology

The Setting for the Study and the Participants

The design of the course in which the study was conducted incorporated a view of pre-service teachers as social constructors of knowledge—as entering the teacher education program with preconceived beliefs (and knowledge) about mathematics and mathematics teaching and learning formed through an apprenticeship of observation during their formal education (Anderson & Piazz, 1996; Ball, 1988; Calderhead & Robson, 1991; Philipp, 2000). Instruction was situated in a university level mathematics content course focused on number theory, sets, and functions among a reform model based upon conceptual rather than a procedural orientation with a focus on meaning making, connections, patterns, justification, and dialogue. Because the relationship between reflection and perturbations are vital to change in teacher beliefs, perplexing classroom experiences were developed which evolved throughout the course of the study. The goal throughout this course was to provide an opportunity for a new kind of "apprenticeship of observation", to develop "teachers' ability and their desire to think seriously, deeply, and continuously about the purposes and consequences of what they do—about the ways in which their curriculum and teaching methods, classroom and school organization, testing and grading procedures, affect purpose and are affected by it" (Silberman, 1970, pg. 472) as well as reflect on their own belief systems. The classroom did not follow a "traditional" format in that lecture was eliminated as the primary form of instruction during classroom learning experiences. Instead, group work and active learning using manipulatives, pictures, and diagrams were the emphasis during class time. A concerted effort was made to establish social norms in the classroom that supported and encouraged discourse, investigation, and questioning. The students' beliefs about mathematics and mathematics teaching and learning were deliberately perturbed through the structure of the class (learning experiences, social norms, etc.). Additionally, writing assignments were incorporated that addressed beliefs (although not always explicitly).

The participants for this study were students from a small 4-year college located in a community comprised of approximately 17,000 people in the southern Midwest region of the United States. All participants were preservice teachers enrolled in a mathematics content course intended for early childhood and elementary majors although there were a few participants taking the course that had other majors such as special education. The research study included two sections of the same mathematics course focused on number theory, sets, and functions totaling 47 students of which 37 chose to be participants: 35 females (95%) and 2 males (5%).

Research Method, Data Collection and Analysis

Teacher action research was a natural research method for this study because it "develops through a *self-reflective spiral* of planning, acting, observing, reflecting, and then replanning, further implementation, observing, and reflecting" (Burnaford, Fischer, & Hobson, 2001, p.43). It allowed for continual action, analysis of data, reflection on the data and the course and opportunities to make adjustments based on those reflections. Data were collected over a period of time of about four months. The majority of the data were collected from participants' writing as part of the course structure via short answer response questions on exams and homework assignments, but other sources included a personal journal, student metaphors, and classroom conversations. The variety and amount of data compiled helped to advance a more complete and accurate sense of the participants' perspectives, experiences, and beliefs.

Despite being personally invested in the research, a teacher researcher can take steps to retain validity and be purposeful in the level of rigor involved with the use of systematic analysis of the data, peer examination and discussion, as well as triangulation (Bartlett & Burton, 2006; Foreman-Peck & Murray, 2008). For this study, a systematic analysis of the data involved a constant comparison method of coding and theming (Strauss & Corbin, 1998); peer examination and discussion entailed corroboration and examination of the findings and themes with a peer;

triangulation necessitated examination of the themes from participants' responses, my reflective journal, and the peer discussions.

Findings

Analysis of the data revealed that participants' perspectives about the nature of mathematics as well as their selves in relation to mathematics changed significantly for many students. It appears that their altered beliefs about mathematics and the culture of the classroom dynamically interacted to affect their mathematical autonomy. Students enjoyed learning mathematics and had greater confidence in their mathematical capabilities. One participant stated

> I think, in general, most of the experiences in this course have enhanced my confidence and enthusiasm for mathematics. Being encouraged to work with out-of-the-box algorithms has expanded my perceived horizons and opened up a new field of interest for me.

Participants' perceptions about the way in which mathematics is "done" as well as their feelings about mathematics changed over the course of a semester in this non-traditional mathematics content course. Many felt more comfortable with the mathematics that they would eventually teach and found a new appreciation for mathematics in general; a few expressed that they even grew to like mathematics. Participants talked about this change in confidence levels extensively; some excerpts from their writings are as follows:

- After this class, math is still not my most liked subject, however it isn't my most disliked either. I do feel a lot more confident in teaching math to students now that I have had this class.
- I think that this course has made me more confident in learning math because it made me realize there was not just one way to find the "answers" to math problems… I am not sure that I will ever really enjoy math but I am not so afraid to take it on now.
- This class has changed my thoughts about math in many ways. Before this class I hated math and I struggled in all my other previous math courses. This class has showed me that math can be enjoyable and that I can do well in this course and not just squeeze by.

Furthermore, after taking a course structured in a non-traditional format that focused on conceptual understanding, meaning-making, answering "why," and working in groups, participants' reported feeling more confidence in their mathematical ability as well as their pedagogical skills to teach mathematics. They spoke about their mathematical and epistemological empowerment related to understanding the mathematics. Participants often linked mathematical empowerment with pedagogical empowerment; they described their newfound confidence to teach others the mathematics content they felt comfortable. For example, one student stated "I know I will be able to teach certain math well because I understand it," and another said "Since taking this course, I have already begun to help my friends and younger siblings with their mathematical endeavors. I think that with more practice I will be an effective teacher with more than just my stronger subjects."

Although, separated initially, these perspective changes into the categories of "Perspectives on Mathematics" and "Perspectives about Self," the intertwining of their beliefs about mathematics, teaching and learning mathematics, and their ability in mathematics form such a dynamic relationship that separating statements into these categories felt like a reduction of sorts. Therefore, below are participant statements about changes in their viewpoints after participation in this non-traditional mathematics content course.

- In the beginning of [this] class I thought she would be a teacher from the textbook like every other kind of math class I have took in the past. But with [this] class it was different. She not only took a little from the standard textbook but from her own ways. She makes us think outside the box… This class has opened my eyes to a new math world, a math world that I will gladly share with my students and colleagues over the years that will come.

- During the course of this past semester I have learned so much. I was apprehensive taking what I felt was a lower level math class again. Because I am not good at math and have never had good math instructors I felt that it would be like every other generic math class I have ever taken; the kind of class where the teacher stands in the front of the classroom and lectures and teaches only from the book and the examples come straight from the book and no further. However this class challenges its students to think outside the box and to get the answer by thinking in a non-traditional sense.

Discussion

The structure, format and experiences designed as part of this non-traditional mathematics course served to empower participants both mathematically and pedagogically. The word *empowerment* encompasses feelings about capability as well as self-confidence (Ernest, 2002). This course eliminated traditional lecture (focused on procedural reasoning) as a primary source of instruction and instead focused on problem-based instruction, student-led solutions, and collaboration time (Gasser, 2011). This study found that the format of class practice affected students in that they reported feeling more confident in their mathematical prowess as well as their ability to teach the mathematical topics covered in the course which supports Anderson and Piazza's findings (1996). For instance, participants stated

- I think, in general, most of the experiences in this course have enhanced my confidence and enthusiasm for mathematics. Being encouraged to work with out-of-the-box algorithms has expanded my perceived horizons and opened up a new field of interest for me.

- I used to view all mathematics very negatively because I was never good at it. However, in here by using visual manipulatives and other methods I was able to better understand mathematics, therefore I can feel more confident about it….because this class gave me a better understanding of mathematics I am able to enjoy it more, instead of being stressed out by it.

While the results of this study are not intended to be generalized, it may be used to inform pre-service teacher preparation programs as well as point to future directions to pursue in research on this topic. After investigating the results of this study, one area for future studies would be to examine how and why pre-service teachers assimilate new ideas to fit existing beliefs rather than accommodate their existing beliefs to internalize new ideas. Moreover, since this study focused on perturbing a variety of mathematical beliefs and found that those perturbed most were seemingly impacted the most, future studies might focus on perturbing specific mathematical beliefs throughout a course to observe the effect on belief structure.

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SPATIAL REASONING IN UNDERGRADUATE MATHEMATICS: A CASE STUDY

Lindsay Prugh
Oklahoma Christian University
lindsay.prugh@oc.edu

The need for spatial thinkers is evident in the lackluster performance of students in mathematics and the lack of interest in spatially-driven fields. Research has linked spatial thinking to problem solving, indicating that spatial thinking skills are necessary for success in mathematics. This embedded case study examined how the inclusion of spatial tasks influenced problem-solving performance, spatial thinking ability, and beliefs of undergraduate mathematics students. Data were collected through quantitative and qualitative instruments. Findings suggest the inclusion of spatial thinking tasks has an influence on students' spatial visualization ability, problem-solving strategies, and beliefs about the relevance of spatial thinking.

Spatial thinking is not only necessary for success in many aspects of daily life, but it is also an essential skill for the STEM fields of Science, Technology, Engineering, and Mathematics, from which many scientific discoveries and progress are made (NRC, 2006). The importance of spatial thinking throughout a child's kindergarten through grade-12 education is emphasized in the geometric standards set forth by the National Council of Teachers of Mathematics (NCTM, 2000). This recommendation is mirrored through the work of the National Research Council (NRC), which asserts that spatial thinking is a learnable skill that should be matriculated throughout a student's educational experience. Spatial activities are a worthwhile investment in the mathematics classroom, since the skill of spatial thinking has been repeatedly linked to problem solving (Battista, 1990; Edens & Potter, 2007; Moses, 1977).

Meaningful mathematics learning is almost always based in spatial imagery. While some forms of mathematical reasoning do not require imagery, the majority of mathematical activities involve a spatial component (Wheatley & Abshire, 2002). But what does it mean to think spatially? Super and Bachrach (1957) describe the skill as the ability to generate, retain, compare, retrieve, manipulate, and transform well-structured mental images. The inclusion of these images through well designed spatial tasks could lead to more effective problem-solving strategies and improved instructional strategies in the classroom. For these changes to be made, present and future students must be given the opportunity to engage in spatial thinking whenever possible, especially in the mathematics classroom.

Overview of the Study

The purpose of this embedded case study was to understand how the inclusion of spatial tasks influenced undergraduate students' spatial visualization ability, problem-solving strategies, and beliefs about spatial thinking. Despite decades of reform, the U.S. still trails economic competitors like Japan (National Center for Education Statistics, 2003). One explanation could be the lackluster ability of U.S. students to think spatially and problem solve with regard to mathematics. As a result, this study examined undergraduate students' abilities and beliefs regarding spatial thinking by addressing the following research questions:

1. How does the integration of spatial activities in an undergraduate mathematics content course impact student spatial ability?
2. In what ways does the integration of spatial reasoning tasks into an undergraduate mathematics content course influence problem-solving strategies?
3. How does the integration of spatial reasoning tasks influence the beliefs on spatial thinking of pre-service elementary teachers?

The participants were 33 undergraduate students enrolled in the researcher's Fall 2011 Survey of Mathematics course. Quantitative data were collected through the Purdue Spatial Visualization Test (PSVT), the Mathematical Processing Instrument (MPI), and the Spatial Thinking Attitude Survey (STAS). Qualitative data were garnered through student-written journal responses, focus group interviews, and observations. A focus group was formed and was comprised of 17 participants who were pre-service elementary education majors. This group met on three separate occasions throughout the study to discuss topics related to the study. The purpose of the focus group was to give deeper insight into the participants' experiences with the study as well as beliefs about spatial thinking.

Implementation began with a description of the study followed by the pre-measures of the PSVT, the MPI and the STAS. During the following eight weeks, a range of daily spatial thinking activities as well as reflective journal prompts were incorporated into classroom practices. For example, one in-class activity asked students to draw the net of a three dimensional figure that they were not allowed to touch. Once the student had completed their drawings, they were asked to share results with the entire class. Activities such as this sparked class discussions, allowing for insightful student observations. These activities would take up approximately 10 minutes of class each day. Then, if further discussion was appropriate, a journal prompt would be given as a follow up. During the final two weeks of the study, the final focus group discussion was held, and post-measures of the PSVT, the MPI, and the STAS were executed and scored. These data were collected and fully analyzed.

Results from the quantitative data were used to determine if the integration of eight weeks of spatial activities resulted in significant differences in scores on the PSVT, the MPI, and individual statements on the STAS. Analysis of the qualitative data—responses to journal prompts, focus group interviews, and observations—was used to examine the influence of the spatial tasks on the perceptions and beliefs about spatial thinking on students and pre-service elementary teachers. Moreover, the same data were used to evaluate how pre-service teachers viewed their own understanding of spatial thinking and its relevance in their daily lives and future classrooms. After the quantitative data had been scored and tested and the qualitative data had been coded and themed, the data were analyzed in its entirety and conclusions were drawn.

Summary of the Findings

Spatial Tasks and Spatial Ability

The first research question investigated the influence of spatial tasks on students' spatial visualization abilities. Bruner (1973) believed children explore new things first through action then through imagery before, finally, using language to describe and comprehend the world around them. Through this reasoning, spatial thinking is a necessary step to learning.

To help investigate the first research question, both qualitative and quantitative data were collected and analyzed. Qualitative analysis on student-written responses and focus group discussions revealed that students believed their spatial thinking abilities could improve with practice. This was encouraging given the fact that 60.6% of the class described themselves as possessing average or below-average ability at best in response to a journal prompt which asked students to describe their ability to think spatially. Using quantitative analysis, the PSVT served as a pre- and post-measure to assess student spatial visualization ability. The PSVT, developed by Guay (1980), was comprised of three parts: Developments, Rotations, and Views. The Developments section (PSVT/DEV) measured spatial structuring; the Rotations section (PSVT/ROT) measured mental rotation ability; while the Views section (PSVT/VIEW) measured spatial perception. Initial assessment of the data revealed an increase in test scores and a decrease in the number of incomplete responses. The DEV, ROT and VIEW sections of the PSVT showed a 27.9%, 33.3% and 60% increase in correct responses from the pre- to post-results, respectively. The individual increases resulted in an overall increase of 38.2% on the total scores from the pre-PSVT to the post-PSVT. Paired samples t-tests were conducted to evaluate whether these changes were significant.

A t-test was performed on each of the three pre- and post-results individually and later on the overall scores. The results of the quantitative analysis revealed a difference in the scores for all areas tested. Notably, these changes were most evident in the overall pre-PSVT scores ($M=14.27$, $SD=6.71$) and the overall post-PSVT scores ($M=19.73$, $SD=7.64$), with $t(32)=6.2$, $p=0.0000006$. Specifically, these results suggest that inclusion of spatial activities for eight weeks increased the students' ability to think spatially, as measured by the PSVT. These results support the NRC's (2006) assertion that spatial thinking can be learned.

Spatial Tasks and Problem Solving

The second area of focus in this study involved spatial thinking and problem solving. Specifically, the second research question sought to identify ways for which the inclusion of spatial tasks influenced mathematical problem-solving strategies. Learning to solve problems is a principal reason for studying mathematics. Problem solving is engaging in a task for which the solution method is not obvious or known in advance, and NCTM (2000) strongly believes this activity is an integral part of mathematics learning. Wu (2004) identified two problem-solving cognitive processes: the factor-analytic approach and the information-processing approach. The former approach is generally empirical, and one factor in this area is visual perception—the concept that spatial/visual aptitude, however strong, will play a role in mathematical problem solving. Several studies support this conjecture (Battista, 1990; Edens & Potter, 2007).

Analysis of the relevant qualitative data collected in this study exposed several themes that involved problem solving. Students felt their problem-solving skills could improve with practice and were important for everyday situations. Participants in this study also believed that spatial thinking was a unique way of thinking. Phrases such as "new way of thinking," "creative thinking," and "spatial mindset" were just a few of the descriptions that surfaced when discussing spatial thinking. Students' perceptions of this "unique" way of thinking did not hinder them from using the skill to aid in problem solving as measured by the MPI.

The MPI was used as a measure to identify the students' preference for solving problems using a visual or non-visual approach. Schematic imagery, as defined by Hegarty and Kozhevnikov (1999), was used when scoring this instrument. Of the 660 possible questions given to the 33 students on the MPI, 55.6% of the questions on the pre-MPI were attempted using a spatial approach. This percentage rose to 62.3% on the post-MPI. The average grade on the pre-MPI to post-MPI changed as well, increasing from 60.6% to 67.7%. As with the PSVT, a t-test for paired samples was used to compare the students' preference for using a visual-spatial approach for problem solving before and after eight weeks of spatial task implementation. A significant difference was revealed in the scores for the pre-MPI ($M=4.76$, $SD=14.16$) and the post-MPI ($M=8.76$, $SD=15.94$) conditions; $t(32)=2.42$, $p=0.021$. These results suggest that inclusion of spatial tasks had an effect on the participants' preference for using a spatial approach when solving problems on the MPI. Specifically, these results suggest that the inclusion of spatial activities increased the preference for using schematic drawings and, therefore, a spatial approach when solving mathematical problems.

This study showed a positive correlation between the PSVT and the MPI, and thereby strengthened the body of existing literature on the relationship between spatial thinking and problem solving. Improvement on one post-measure typically indicated improvement on the other. Through journal responses and discussions, participants stated they had "more confidence" when taking the MPI the second time. Fisher (2005) explained that "visual expression provides a means of formulating and solving problems" (p.16), so improvement on these two instruments makes sense. Based on these results, it is apparent that exercises in spatial thinking affect spatial ability as well as one's preference for using a spatial approach when problem solving in mathematics. A change in students' beliefs seems like a logical extension of the change in students' confidence and ability.

Spatial Thinking and Pre-service Teachers' Beliefs

In addition to examining the influence spatial tasks had on ability, this study explored the impact of spatial activities on beliefs of pre-service elementary teachers. The beliefs of pre-service teachers are an important component of spatial thinking and problem solving, since research has shown that teachers who are more confident in their own spatial abilities are more likely to use such strategies in their classrooms (Battista, 1990; Presmeg, 1986). Qualitative analysis on the STAS showed considerable change in teacher beliefs concerning the usefulness of spatial thinking outside of mathematics.

The STAS, developed by Hanlon (2009), was a 15-question, five-point, Likert-type survey that partially focused on measuring beliefs regarding spatial thinking. Notably, question number four asked if "spatial thinking skills are useful in other areas besides mathematics." Seven of the 17 students answered "Disagree" or "Strongly Disagree" on the pre-STAS. This number dropped to only two the post-STAS. A t-test for paired samples was used to measure for significant change in responses on all 14 questions of the STAS. A significant change was found for the following seven areas of spatial thinking and geometrical drawing: Spatial thinking skills are important for students to be successful at the elementary school level; I am sure that I can improve my spatial thinking abilities; Spatial thinking skills are useful in other areas besides mathematics; Spatial thinking skills can be developed; I will incorporate spatial thinking activities into the classroom; I can see spatial thinking in many aspects of my daily life; I am confident that I can draw geometric shapes accurately.

These results indicate that eight weeks of spatial tasks changed the beliefs of pre-service elementary teachers. Specifically, after the implementation of spatial activities, the participants were more likely to believe that spatial skills are malleable, useful outside the mathematics classroom, and worthy of inclusion in future curricula. The exercises students experienced throughout the eight weeks of implementation promoted understanding of spatial concepts and allowed students the opportunity to identify other areas where spatial skills are useful.

Concluding Comments

The need for practiced spatial thinkers is evident in the growing concern over performance of U.S. students in mathematics as well as lack of interest in the spatially driven fields of STEM. In addition to this need, spatial thinking is a beneficial skill that reaches beyond the STEM fields, as good problem-solving techniques are valuable for everyday life. Since spatial thinking is related to problem solving, and problem solving is important in many facets of life, spatial thinking should be a skill that is fostered and encouraged within the classroom.

According to NCTM (2000), problem solving is an integral part of all mathematics learning. Therefore, students must be given the opportunity to foster this skill from the beginning to the conclusion of their educational experience. Thankfully, research, including this study, has shown that this vital skill can be improved as late as post-secondary school. Spatial skills need to be intentionally nurtured if educators desire to give students a global competitive edge and help students develop an effective arsenal of strategies to problem solve. While this skill is not explicitly tested by state exams, the benefits of honing spatial skills will pay off long after the final bells of a classroom have rung. If the purpose of education is to create productive citizens to advance our way of life, then spatial thinking must be incorporated into the classroom. To do this, we must first equip our future teachers.

The role of research concentrating on pre-service teachers' spatial thinking and spatial ability needs to be a priority if change is desired. The spatial thinking and beliefs surrounding spatial thinking of pre-service educators is a critical component to the likelihood of this skill being fostered in future mathematics classrooms. The spotlight is now on teacher education programs, because pre-service teachers must first be proficient spatial thinkers before they are able to infuse this skill into their own teaching methods. Mathematics courses—especially those required for education majors—should be used as a fundamental piece to this design. In conclusion, for change to occur, inclusion of spatial thinking and spatial thinking activities must permeate the mathematics classrooms and teacher education programs of today and tomorrow.

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STUDENT CONCEPTIONS OF "BEST" SAMPLING METHODS: INCREASING KNOWLEDGE OF CONTENT AND STUDENTS (KCS) IN STATISTICS CLASSROOMS

Jeremy F Strayer
Middle Tennessee State University
Jeremy.strayer@mtsu.edu

Many students enter statistics courses with misconceptions regarding sampling methods. This article presents results of a study that: (1) provides an empirical analysis of the nature of two common student misconceptions and (2) introduces a hypothetical learning trajectory for strengthening students' understanding of sampling methods.

Virtually every inferential statistical method taught in introductory statistics courses assume the data comes from a simple random sample. Unfortunately, most students (and many textbook authors) confuse simple random samples with other types of samples (e.g. cluster samples) (Alf & Lohr, 2007). Not only do students confuse different types of samples, some have deep-seated misconceptions regarding what makes for a good sample. In particular, students often believe that: (1) sample size must be relatively large (e.g., half of the population) in order to provide reliable information about the population under study and (2) convenience samples are representative and/or random. Since most introductory statistics courses share the goal of facilitating students' proficiency in conducting inferential statistical studies, it is essential for instructors of these courses to possess a deep understanding of student thinking in order to support student learning of appropriate sampling methods.

Theoretical Framework

Mathematics educators have long understood the importance of attending to students' errors and misconceptions in the learning process. Over the past three decades researchers have made inroads into how errors and misconceptions are investigated. Initially, educators focused on errors in student procedural skill, such as 32 – 17 = 25. These misconceptions were understood as semantically meaningful (to the student) deviations from correct procedure, and termed "mal-rules" or "buggy algorithms" (Brown & Burton, 1978). Efforts were made to help students correct their misconceptions by identifying them, giving meaningful feedback, and providing repeated corrective activities using technology (Burton, 1982). One early study of this type in statistics education investigated student misconceptions of means and variances (Mevarech, 1983). This study provided evidence to support the hypothesis that students' errors exist because they believe the operation of averaging two numbers possesses the group properties of closure, associativity, identity, and inverses. Mevarech concludes that students' misconceptions cannot be corrected simply by repeatedly demonstrating the correct procedure in a lecture and discussion setting. Rather, students must have the opportunity to receive feedback and engage in corrective activities.

Li & Li (2008) note a shift in focus for mathematics education misconception studies from a focus on deficiencies in reasoning (Brown & Burton, 1978) to a focus on the learning process itself. They suggest that advances in science education misconception research have paved the way for mathematics education researchers to better understand why student misconceptions are so resistant to change. Specifically, studies suggest that students' initial mathematical understandings often exhibit process-like thinking (e.g., conceptual understanding of number begins with counting). As students' mathematical learning progresses, their understanding begins to exhibit object-like thinking (e.g., a more abstract notion of number as a mathematical object). Li & Li challenge researchers to develop theories of mathematical misconceptions that take this shift in understanding into account. Perhaps misconceptions formed during process-like thinking persist into students' object-like conceptualizations of mathematics.

Even as we progress in our understanding of what underlies student misconceptions of mathematics, just knowing about misconceptions is not enough. In an effort to answer Shulman's (1986) charge to develop a coherent framework of the knowledge necessary for effective teaching, Ball, Thames, & Phelps (2008) propose the domains of mathematical knowledge for teaching (MKT) (Figure 1). According to the primary MKT domains, successful teachers must possess different types of content knowledge. First, teachers must know the content they are teaching to students in a way that most people knowledgeable in the content area do. This knowledge is referred to as *common content knowledge* (CCK). In addition to CCK, teachers must also have knowledge of the content that enables them to understand the multiple ways students interact with that content as they are learning. This type of knowledge is *specialized content knowledge* (SCK) because it consists of unique knowledge of the content that teachers must possess but a typical person knowledgeable of the content area would likely not.

The MKT framework divides content knowledge closely related to the teaching and learning process (pedagogical content knowledge) into two primary domains: *knowledge of content and students* (KCS) and *knowledge of content and teaching* (KCT). KCS pertains to the ways that students are likely to interact with and make sense of the content. This domain addresses knowledge of misconceptions, common mistakes, and common points of confusion as well as topics that students find easy to learn. KCT is the knowledge of content as it relates to instruction. It addresses effective sequencing of content during the completion of learning tasks as well as the content knowledge necessary for choosing appropriate examples.

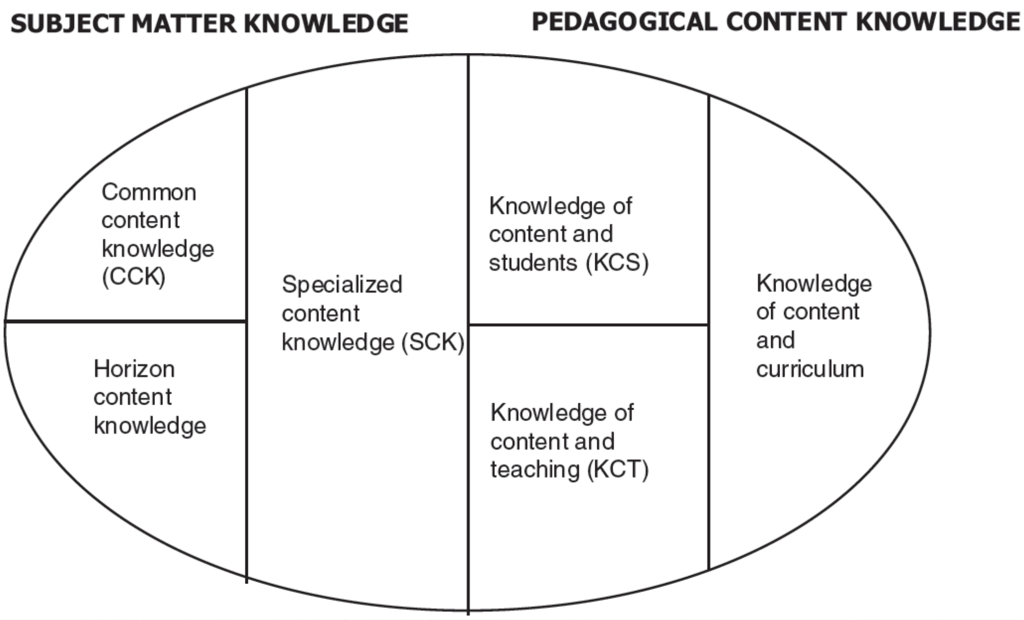

Figure 1. Domains of Mathematical Knowledge for Teaching (Ball, Thames, & Phelps, 2008)

The research reported in this paper investigates pedagogical content knowledge for statistics teachers. There is widespread agreement that teaching mathematics and teaching statistics, while certainly related, are distinctly different enterprises. In fact, Groth (2007) has proposed a framework for *statistical knowledge for teaching*. This framework describes the unique ways that statistics as the subject matter (the left side of Figure 1) influence both CCK and SCK. Drawing on the notion that *context* distinguishes statistics from mathematics, Groth makes explicit the ways that both mathematical knowledge and nonmathematical knowledge must be activated within CCK and SCK when teaching statistics.

This present study focuses on the right side of Figure 1 in general, and KCS in particular. Specifically, I seek to deepen our understanding of the ways students understand and think about what makes a sample "good" in a simple statistical study. When considering the evidence from this study, it is important that we follow Groth's lead and consider the types of nonmathematical knowledge that students may be activating when conceptualizing appropriate sampling methods.

While many studies have investigated students' conceptions and misconceptions of average, variability, distributions, sampling distributions, and correlation (Shaughnessy, 2007), few specifically investigate students' conceptualizations of sampling methods. To be sure, many statistics education teaching resources contain quality tasks that challenge students to develop a sound understanding of issues involved in selecting representative samples (Franklin et al., 2007; Rossman & Chance, 2008; Warton, 2007); however, it appears that most of these resources were developed using authors' experience and expertise in teaching statistics rather than empirical studies of student thinking.

For example, Warton (2007) presents a task that requires students to estimate the size of their vocabulary using a dictionary. In the process of completing this task, students must select a representative sample of words from the dictionary – a nontrivial subtask. The article mentions that it is important to discuss potential sampling methods (highlighting their strengths and weaknesses) and reports that students commonly ask questions such as "But how many samples should I take?", "How do I decide how precise I want my estimate to be?" and "Why not use a systematic sample rather than a random sample?" While these sample questions provide helpful insight regarding what to expect when this task is implemented with students, an analysis of students' conceptualization of the issues involved in sampling is not given. An analysis of this sort would provide helpful information to teachers who wish to implement this and other tasks focused on sampling methods.

Methods

This study was conducted to better understand student ways of thinking related to sampling methods misconceptions I have observed in 14 years of teaching introductory statistics courses. One prevalent misconception is the belief that samples must be very large (e.g. half the size of the population) to be representative. A second common misconception is the belief that a convenience sample is an acceptable sampling method for gathering data useful for drawing sound inferences about the population of interest.

Participants in this study included 22 members of an introductory statistics class. These students completed a pre-test for the course in which they were asked to answer a question regarding the best way to take a sample from all students at a university in order to gauge the opinion of the student body. This question is designed to reflect a situation where a simple random sample is required in order to analyze the data with standard inferential statistical methods taught in an introductory statistics course. The question is a modified version of a question found at the NSF funded ARTIST (Assessment Resource Tools for Improving Statistical Thinking) website (Garfield, 2006) and is as follows:

> Four students at XYZ University (Ashley, Jake, Adam, and Keisha) conduct surveys to gauge the opinion of the student body on various political issues. The student body is 30,000 students. **Ashley** got the names of all students at XYZU, put them on pieces of paper in a large plastic container, mixed them well, and chose 120 students to ask. **Jake** asked 50 students at a meeting of the computer gaming club. **Adam** asked all 8,293 students who are sophomores. **Keisha** set up a booth outside of the student union and asked people passing by to fill out a survey. She stopped collecting surveys when she got 120 students to complete them. Discuss the benefits and limitations of each person's sampling method. Which method do you think is best?

At the beginning of the semester, students provided written feedback to this question and all 22 students explained their thinking during a video recorded semi-structured interview. Subsequently, students learned standard introductory statistics topics including sampling distributions, confidence intervals, and hypothesis testing through ANOVA and linear regression. Lessons that targeted issues in the above modified ARTIST question included one task that utilized the TI-Nspire and empirical sampling distributions to investigate how large random samples need to be for a sample mean or proportion to provide a good estimate of the population mean or proportion (Strayer, in press). Another task used decks of cards to simulate the problems with taking convenience samples (see Appendix). At the end of the semester, the 22 participants completed the same assessment as a post-test and were again interviewed to investigate their thinking on the above question. Student answers to the question were recorded and the video interviews were transcribed. An analysis of data from the transcripts and written responses was conducted using qualitative grounded theory methods of open coding, memo writing, axial coding, and theory construction.

Results

Students' choices for their preferred sampling plans at the beginning and end of the semester are shown in Table 1. In the pre-test, 59% of the students chose the correct answer of Ashley, while 95% chose Ashley on the post-test. Incorrect answers were split evenly on the pre-test with 18% choosing Adam (large sample misconception) and 18% choosing Keisha (convenience sample misconception). All four of the students who chose Adam on the pre-test correctly chose Ashley on the post-test, and three of the four students who chose Keisha on the pre-test chose Ashley on the post-test.

Table 1

Number of Students Choosing Each Sampling Plan

	Ashley	Jake	Adam	Keisha
Pre-test	13	0	4	4
Post-test	21	0	0	1

Students' reasons for choosing their preferred sampling plan varied widely. An analysis of the pre and post interview data revealed a three tiered structure to the misconceptions the students had regarding appropriate sampling methods. At the lowest level, students had three distinct general misconceptions. Some students placed an inordinate amount of attention on whether or not the sampling plan was **efficient**. If a sampling plan was too difficult to carry out, it was dismissed. For instance, some students said it was too difficult for Ashley to fit all the names of the students in a hat, so she shouldn't do it. While efficiency should be attended to (for instance, it would not be advisable to conduct a census in this study), some students had an underdeveloped sense of efficiency that interfered with their understanding of the benefits of choosing a random sample in this case. A second misconception at this lower level is the belief that researchers must have an extremely **large sample** in order to produce reliable results. Finally, many students held the belief that it is critical to the success of the study for people in the sample to **care** about the survey topic.

Students with a more developed understanding of issues involved in sampling recognized the benefits of having a random sample, but they possessed misconceptions of what makes for a random sample. Some students felt that if the sample was **diverse**, then it was random. So long as there was a good mix of people, it was random. Others expressed that if there was a **haphazard** way of selecting sample participants ("no rhyme or reason"), then the sample was random. Finally, some students felt that if the sample was a volunteer sample, then it was random. Since the sample participants "randomly" came up to the researcher and were **not chosen** by the researcher, the sample was random. In part, these misconceptions likely stem from a colloquial understanding of random as a surprising or unexpected event.

At the highest level of reasoning about the sampling methods, students understood the importance of having a sample that is representative of the larger population. However, misconceptions of what makes a sample representative persisted. For example, some students perceived **diversity** in the sample as a sign that it is representative. Other students held the belief that as long as the entire population was **available** to be chosen for the sample, then the sample was representative. Very few students recognized the need for all members of the population to have an **equal chance** of being a part of the sample in order for the sample to be truly random (a simple random sample) with a high probability of being representative.

A majority of students in this study showed progress toward deepening their statistical reasoning from pre to post interview. This observed progress followed paths along the three-tiered structure described above, suggesting a hypothetical learning path (Clements & Sarama, 2004) along which students may tend to progress as they develop the necessary understanding of what it means to have a good sampling plan.

Conclusion

Students can acknowledge the importance of random sampling in a statistical study yet have a limited understanding of what this means. Indeed, understanding can be confounded by the fact that it is often difficult or impossible to conduct true random samples of populations in specific contexts (e.g. the common context of predicting elections). In the midst of this messiness, how can teachers help students conceptualize appropriate sampling methods for a research study? This project sought to answer this question by developing KCS with regard to how students understand sampling methods ideas. The results of this study suggest a hypothetical learning trajectory along which students may travel as they think through critical issues associated with choosing representative random samples from populations.

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APPENDIX

To use two decks of cards to simulate the shortcomings of convenience sampling, separate the decks into black and red cards. Place all the black cards on top of the red ones and without showing the color of the cards to the class, tell the students that we are interested in taking a sample from this stack (the population) using a similar sampling plan as Keisha. The goal of the study is to determine the percentage of black cards in the stack. Have a student volunteer to be Keisha and ask, "Did Keisha have a choice of which students came by the student union the day she took her sample?" Once there is agreement that Keisha had no choice of who walked by, select cards from the top of the deck so that all of them or nearly all of them are black and (without showing the color) give them to "Keisha" saying that these are the "cards" that "walk by" her, so these are the cards "Keisha" has to choose from. Now, "Keisha" gets to choose her sample from this stack however she wants, but she cannot look at the color until she has chosen the sample. After "Keisha" takes the sample, have the student look at the color of the cards in the sample and predict what percentage of the stack is black. Since the sample will be nearly 100% black, "Keisha" will predict that nearly 100% of the larger stack is black as well. At this point show the class the colors of the cards in the entire stack (i.e. population). A class discussion of the shortcomings of convenience sampling can be lead using the following key questions: 1) What aspects of the sampling plan did Keisha have control over and what aspects of the sampling plan did Keisha not have control over? 2) In what ways does Keisha's plan *feel* random? and 3) How is Keisha's plan deceiving, if the goal is to get a sample that represents the population well?

PREPARING K-10 TEACHERS THROUGH COMMON CORE FOR REASONING AND SENSE MAKING[1]

Jonathan Bostic
Bowling Green State University
Bosticj@bgsu.edu

Gabriel Matney
Bowling Green State University
Gmatney@bgsu.edu

There looms an uncertainty about the Common Core State Standards for mathematics for many teachers. Teachers have indicated that they want professional development (PD) focused on learning about the new standards (Bostic & Matney, in press). This manuscript describes PD programs for K-10 mathematics teachers and offers results from one activity aimed to help teachers unpack the Standards for Mathematical Practice (SFMP). Four major themes arose from interpretive analysis of teachers' perceptions of the SFMP. These findings suggest (1) the PD supported teachers to make sense of the SFMP and (2) teachers may have misperceptions about the SFMP that require further PD.

Standards for Mathematical Practice

Mathematics instruction in the era of Common Core State Standards for mathematics (CCSSM; Council of Chief State School Officers [CCSSO], 2010) will require teachers to reevaluate their current instruction (National Council of Teachers of Mathematics [NCTM], 2010). The CCSSM built upon decades of work "to define the mathematics that students need to know and be able to do" (NCTM, 2010, p. ix). A critical element of the CCSSM is the overarching emphasis given to the Standards for Mathematical Practice (SFMP). The SFMP offer descriptions of behaviors that students should demonstrate while learning mathematics. The SFMP were created from two foundational texts: *Principles and Standards for School Mathematics* (NCTM, 2000) and *Adding it Up* (Kilpatrick, Swafford, & Findell, 2001). NCTM's (2000) process standards are problem solving, reasoning and proof, communication, connections, and representation. The notion of mathematical proficiency includes conceptual understanding, procedural fluency, adaptive reasoning, strategic competence, and productive disposition (Kilpatrick et al, 2001). Elements from the process standards and the mathematical proficiency are evident in the SFMP. Unfortunately, these ideas are not evident in every classroom. Thus, professional development must be designed to enhance teachers' understanding of the SFMP and ways to encourage these behaviors in their mathematics classrooms. These behaviors are not

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isolated and often occur in tandem with one another because they are interrelated behaviors (CCSSO, 2010). For example, making sense of problems and looking for mathematical structure are likely to occur during a problem-solving session. In order for students to elicit behaviors indicative of the SFMP, teachers must design and enact instruction that allow students to wrestle with mathematics content and its applications in an environment that supports and sustains meaningful engagement with mathematics.

There is no prescribed set curriculum or pathway for teachers to encourage these behaviors in their students; however, worthwhile tasks and mathematical discourse provide a vehicle for supporting students' mathematical thinking (NCTM, 2007). Video analyses of USA teachers' instruction indicates that generally speaking, teachers are not promoting the process standards or mathematical proficiency (Hiebert et al., 2005), much less the SFMP. Hence, mathematics teacher educators should provide professional development that assists K-12 mathematics teachers' understandings of the student behaviors found in the SFMP and how those behaviors can be promoted through their instruction. The purpose of this paper is to discuss K-10 mathematics teachers' perceptions about the SFMP.

Professional Development

Teachers need professional development during the transition to the CCSSM. This PD "will require practical, intensive, and ongoing professional learning - no one-off 'spray and pray' training" (Hirsh, 2012). An underlying goal of most professional development is to enhance teachers' understanding of content, pedagogy, or content-focused pedagogy. Results from a national sample of more than 1,000 mathematics and science teachers indicated that three factors are most likely to influence teachers' practices: (1) connection to teachers' prior experiences, (2) alignment with standards, and (3) opportunities to share ideas with other teachers (Garet, Porter, Desimoney, Birman, & Yoon, 2001). Engaging teachers with mathematics content in a way that fosters hands-on learning and finding ways to integrate PD activities into a teachers' daily life led to longer lasting positive instructional outcomes (Garet et al., 2001). Thus, mathematics teacher educators ought to focus on these factors to promote coherent PD.

A metaanalyis of PD suggests that there are some key features to designing effective inservice teacher education (Guskey & Yoon, 2009). First, workshops and summer institutes that focus on implementing research-based instructional practices, active learning, and opportunities to adapt these practices in the classroom were highly correlated with positive student outcomes. Second, PD led by university faculty or consultants outside of a school district tended to foster more positive outcomes than PD delivered by school personnel. Third, purposefully structured and directed PD that focused on content, pedagogy, or both and lasted more than 30 contact hours was positively associated with improving students' outcomes. Fourth, activities that encourage teachers to adapt a variety of practices to a content area are better than encouraging a set of "best practices". That is, teachers ought to learn how to adapt to novel situations and use a variety of teaching tools. Fifth, effective PD supports teachers' content or pedagogical content knowledge and the PD is situated in knowledge drawn from how students learn. Finally, effective PD includes follow-up activities after the main professional development. With these features in mind for designing successful PD, two PD projects were conducted in a Midwestern state in an effort to prepare teachers to implement the CCSSM. This manuscript provides insight into one research question stemming from an activity conducted during the projects: What are teachers' perceptions of the SFMP? Teachers' perceptions about the SFMP will help mathematics teacher educators design and implement PD intended to focus on the SFMP.

Method

Context of the Professional Development

This manuscript synthesizes results of an activity that occurred during two grant-funded yearlong projects. Each author was a project director for one PD program and co-primary investigator on the other. Teachers met four times for four-and-a-half hour sessions between March – April 2012. Next, participants and instructors met for eight 8-hour days during June 2012. Finally, teachers met twice face-to-face for four-and-a-half hour sessions between August – October 2012. Instructors also provided numerous online assignments and facilitated online interactions between March – October to support teachers' understanding of the SFMP. Since the teachers performed the same SFMP unpacking activity in each of their respective PD programs over the course of two meetings, data from the two programs are combined.

Every program included teachers from a high-needs district (i.e., more than 20% of students come from families below the poverty line and a large percentage of teachers are teaching out of their licensed field). Generally speaking, the aims of the PD projects included (1) making sense of the SFMP, (2) exploring inquiry through worthwhile tasks, mathematical discourse, and appropriate learning environments, and (3) implementing classroom-based tasks that aligned with the CCSSM. Teachers read and discussed chapters from NCTM books and completed various assignments including reflective journaling, writing, enacting, and reflecting on CCSSM-aligned mathematics lessons, and solving mathematics problems.

Participants

One grant-funded project served 23 grades K-5 mathematics teachers while the other grant-funded project supported 23 grades 5-10 mathematics teachers. The K-5 and grade 5-10 teachers met separately due to geographic constraints. Teachers came from urban, suburban, and rural school districts. Teachers consented to being video recorded during the PD.

Procedures

Unpacking the SFMP activity. The teachers were given an activity to make sense of the SFMP during the third and fourth meeting dates of their respective PD meetings during Spring 2012. They were assembled into groups of two to four participants. Groups were strategically made so that teachers shared ideas with others teaching similar grade levels but located in different school districts. Elementary teachers were organized into grades K-2 and grades 3-5. Middle level and secondary teachers were organized into grades 5-7 and grade 8-HS teachers. Teachers were asked to describe the SFMP in a manner that the following three kinds of people might understand: (1) a child in their respective grade levels, (2) a parent or administrator, and (3) a fellow teacher of mathematics. After creating the description, they were expected to role-play a classroom scenario depicting an aspect of the SFMP provided to them. Groups were encouraged to behave as the teacher and students or role-play a scenario with only students. Finally, the rest of the teachers shared whether and/or to what degree the SFMP was evident in the skit. The instructors synthesized teachers' ideas during this final share time. Teachers' descriptions and role-plays were videotaped and later transcribed.

Data Collection and Analysis

The unpacking of the SFMP activity was videotaped. Videotapes provided adequate visual and audio evidence of the interactions, cues, writing, technology and expressions used during the role-play and ensuing conversations. Data were analyzed using interpretive analysis (Hatch, 2002). First, videos of the unpacking activity were transcribed. A matrix was created to organize ideas during the coding process. Each SFMP was ascribed a column and each group of teachers was assigned a row. Second, three coders (two mathematics education faculty and one graduate research assistant) watched the videotapes and read transcripts simultaneously to familiarize themselves with the data. Videotapes were paused after each role-play to allow the coders to discuss the activity and share ideas. Initial ideas about each group's role-play were recorded as memos to reflect on during iterative and subsequent analyses. Third, the coders reviewed the memos within the matrix for overarching themes that transcended across groups, grade levels, and/or SFMP. Fourth, themes were reexamined for substantial evidence and a paucity or lack of evidence. Themes were retained when there was substantial evidence from the videotapes and/or transcripts. The fifth and final stage in the process was to rewrite the themes as complete sentences and consider viable representations to convey the coders' interpretations of teachers' perceptions of SFMP through the activity.

Results

Four themes were revealed as a result of the interpretive analyses. The first theme was that there is a lack of evidence that teachers understand SFMP #1. Teachers' role-playing activities provided little evidence of any behavior described in this standard. For example, the high school group of teachers role-played a scenario in which students worked with system of equations using a graphing calculator. Language within SFMP #1 stated that "older students might, depending on the context of the problem…change the viewing window on their graphing calculator to get the information they need" (CCSSO, 2010, p.6). Their task was meant to be an exercise rather than a problem. They interpreted expanding the graphing window to examine a system of equations as evidence of this standard. Unfortunately, these high school teachers perceived that merely changing the viewing window while working on an exercise is sufficient evidence of SFMP #1. A critical component to demonstrating SFMP #1 is providing students with a worthwhile task that is problematic.

A second theme that emerged was that the norm of classroom environments impact the depth and quality of the SFMP that may be exhibited. For example, the group of intermediate elementary teachers role-played SFMP #4: Model with Mathematics. Specific to this role-play, the teachers enacted norms such as (a) students are expected to discuss the effectiveness of the model and its representation, (b) students are expected to discuss the mathematics within the model, and (c) reason quantitatively as described in SFMP #2. Other sociomathematical and mathematical norms were displayed in other groups' role-plays such as (a) commenting on others' ideas rather than the person and (b) exploring and discussing alternative strategies, models, and solutions. An example of teachers' misperceptions was evident in middle school teachers' role-play of SFMP #4. A participant acted as the classroom teacher while two other group members behaved as students. The teacher offered a verbal exercise and then asked students to solve it. The teacher used an initiate-respond-evaluate (IRE) discourse pattern (see Durkin, 1978-1979) and proceeded to focus students' inquiries into finding a solution to the task instead of creating and evaluating the model. The teacher positioned himself as the authority and did not take up students' ideas and explore them. Instead, he perceived his goal was to find the solution rather than explore the appropriateness of the model. This role-play and others made it clear that classroom norms impact students' ability to adequately demonstrate practice standards.

A third and important theme was that teachers struggled with the notion that the SFMP are written for students to demonstrate. This is clearly evident in the language of the SFMP because every standard begins with "mathematically proficient students…" (CCSSM, 2010, pp. 6-8). Thus, students should be the ones showing these behaviors. The teacher is the facilitator in the classroom who creates a context for students to engage in these mathematical behaviors. The videos of the role-play consistently showed that teachers struggled with determining what it is students should exhibit as evidence of the behaviors in the practice standards. The intermediate group given SFMP #7 employed the circles and stars activity (Burns, 1991) to model thinking about multiplication. That is, does *a* times *b* describe *a* groups of *b* items or *a* items collected into *b* groups? The teacher in the role-play showed students how to group the items and did not allow students to wrestle with this mathematics question. Similarly, the middle school group role-played an example of a verbal exercise given to students as an example of SFMP #7 (i.e., How many M&Ms are needed if there are 15 students in a class and each student should receive nine M&Ms?). Again, the teacher led the instruction using an IRE format and directed students' thinking with guiding questions. Students were not provided with a problematic task much less time to wrestle with it, and were not expected to demonstrate the behaviors indicated in the SFMP. Teachers often perceived their role was to demonstrate the behaviors and encourage students to notice how the teacher behaved mathematically.

The fourth and final theme revolved around teachers' mathematics experiences. That is, the kind of mathematics teaching involving the CCSSM, specifically the SFMP, were not experiences the teachers had as students. It was difficult for teachers to interpret the SFMP and implement them in their role-play activity as the CCSSM authors might desire. Teachers shared during conversations following the role-play activity that their mathematical experiences in school were composed of completing exercises and engaging with the teacher in an IRE format. These four themes provide insight into teachers' perceptions of the SFMP and also point to features that mathematics teacher educators should consider when enacting PD focused on the SFMP.

Implications

This PD activity and its results ought to impact how teacher educators design and implement professional development. One issue is that teachers have not personally experienced mathematics learning behaviors like those described in the SFMP. The SFMP do not dictate curriculum or teaching but they do provide ideas for the types of behaviors that mathematically proficient students ought to exhibit during classroom instruction. If teachers are expected to encourage these behaviors in their students then they may need to experience mathematics instruction that allows them to engage in these behaviors. PD may help mathematics teachers at all grade levels make sense of mathematics instruction that supports students' appropriate mathematical behaviors.

Limitations

This study has some limitations that impact the results and conclusions. First, the intrepretivist approach to analyzing data was selected because it allowed the coders to make sense of the data and draw logical conclusions. It is possible that another coder or set of coders might draw different conclusions. Qualitative approaches allow researchers to draw on their lenses and frames of reference to make sense of experiences in the world. The results offered here are not generalizable to all teachers and are particular to this set of teachers. A second limitation is the sample of teachers. These teachers volunteered to participate in mathematics professional development, which is an indicator of motivation to improve oneself. Our themes might differ if the sample included teachers who were less motivated to do PD. Furthermore, teachers with different prior (i.e., mathematics and mathematics content) knowledge and experiences teaching in other contexts (e.g., metropolitan districts) might lead to different results.

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PERCEPTIONS OF THE STANDARDS OF MATHEMATICAL PRACTICES AND PLANS FOR IMPLEMENTATION

Travis A. Olson
University of Nevada, Las Vegas
travis.olson@unlv.edu

Melfried Olson
University of Hawaii
melfried@hawaii.edu

Stephanie Capen
University of Hawaii
stephaniecapen@gmail.com

We solicited feedback from over 20 teachers on two questions related to each Standard for Mathematical Practice. Specifically, after reading each standard, we asked the teachers: 1) Name one or two things that caught your eye as you read the standard, and 2) What is one way you are, or plan on being, more intentional about this standard in your teaching? In our paper we discuss the responses regarding ideas that stand out for teachers per standard and classify their plans for being more intentional about the standards in their teaching.

The Common Core State Standards of Mathematics (CCSSM) (National Governors Association Center for Best Practices, Council of Chief State School Officers [CCSSI], 2010) essentially are now established as the mathematics curriculum framework for the United States. This framework describes well-articulated standards of mathematical content delineated for students at each grade level in grades K – 8 (K – 12 in the Integrated Pathway) as well as by subjects (e.g. Algebra I, Geometry) in the Traditional Pathway. This framework also articulates eight Standards for Mathematical Practice (SMP) described as behaviors, attitudes, skills, or attributions that students should possess. In introducing the SMP, the authors of the CCSSM state, “The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should develop in their students” (CCSSI, 2010, p.6). Each SMP begins with the words “Mathematically proficient students” and goes on to articulate what students should be doing to develop the necessary proficiency related to that SMP. Both the standards for mathematical content and standards for mathematical practice are being implemented in most states with high expectations for student achievement and teacher implementation.

While the SMP describe proficiencies students should develop, little is said how teachers should to develop these proficiencies with their students. The standards documents published by the National Council of Teachers of Mathematics (NCTM) (1989, 1991, 2000) address teaching of mathematics and serve as one basis for the CCSSM SMP (CCSSI, 2010). Researchers have investigated the degree to which teachers are aware of the various NCTM standards documents, as well as the alignment between the standards and teachers' beliefs (LaBerge, Sons, & Zollman, 1999; Markward, 1996; Mudge 1993; Perrin, 2012; Zollman & Mason, 1992). These studies indicate a range of teachers' familiarity and awareness of the NCTM standards documents. They also highlight varying degrees of alignment between teachers' beliefs and the philosophies implied by the standards. As highlighted by Heck, et al. (2011), however, given the recent publication of the CCSSM, similar studies are likely only emerging with regard to the CCSSM. Consequently, questions arise as to what mathematics teachers obtain from initial readings of the SMP regarding how they believe they can implement, or are implementing the ideas therein. Specifically, the research reported in this paper addresses the following two research questions:

1) When teachers initially read the SMP, what do they report as noteworthy?
2) When teachers initially read the SMP, what aspects of each standard do teachers identify as influencing their intentions to address the SMP in their instruction?

Framework and Related Literature

Our formulation of the research questions and associated analysis of teacher's responses were framed primarily by timely policy oriented discussions regarding research on the CCSSM. In particular, a recent report of a national meeting to examine the impact of the CCSSM, and to outline a priority research agenda for understanding their influence identified five research areas was published (Heck et al., 2011). The Priority Case Study Focus #5 of the document, Teacher responses to the CCSSM, is stated as follows:

> Since teachers' knowledge, interpretations, self-efficacy, beliefs, dispositions, and skill, as well as their specific intentions and plans, affect what transpires in classrooms, it is critical to understand how teachers respond to the CCSSM, and what kinds of classroom learning opportunities for their students result. (Heck et al., p. 13)

Within this priority case study focus, Heck et al. (2011) articulate four broad areas of questions that should provide a focus for studies undertaken to investigate teacher responses to the CCSSM. Our paper focuses on the questions used to describe and explain studies that address the second set of such questions, "What implications do teachers see for their mathematics instruction? What aspects of their mathematics instruction do they see as validated by the CCSSM, and what aspects do they consider in need of change based on the CCSSM?" (p.13).

Additionally, the research reported in this paper addresses the call by Heck et al. (2011) for research "status studies" that report the current status of CCSSM adoption and implementation.

Our research and analysis should be considered as initial baseline data. That is, our analysis provides a status of how teachers interpret the SMP as they begin CCSSM implementation.

Methodology

Twenty-three teachers in two separate in-service experiences were surveyed for this study. The teachers were asked to first read the SMP, and then respond to two prompts per standard: 1. Name one or two things that caught your eye as you read the standard, 2. What is one way you are, or plan on being, more intentional about this standard in your teaching?

In one instance, 7 in-service middle-grades (Grades 6-8) teachers were surveyed at the start of a professional development day. In a separate instance, 16 in-service Grade 1 through collegiate teachers were surveyed at the start of a masters-level university mathematics education course. The prevailing sentiment of the respondents was that no respondent had more than briefly skimmed the SMP prior to the survey. Consequently, their “familiarity” with the SMP was essentially categorized as “not read” (as defined by Perrin, 2012). Thus, all of the responses to the survey were considered to reflect in-service teachers’ perceptions after their *initial* reading of each SMP.

Although the teachers were instructed to read each SMP in its entirety, for brevity Table 1 lists only the title of the SMP that teachers read and for which they provided their responses.

Table 1

Standards for Mathematical Practices

Standard	Title
1	Make sense of problems and persevere in solving them
2	Reason abstractly and quantitatively
3	Construct viable arguments and critique the reasoning of others
4	Model with mathematics
5	Use appropriate tools strategically
6	Attend to precision
7	Look for and make use of structure
8	Look for and express regularity in repeated reasoning

There is variation in the grade levels taught, but 17 of the 23 teachers teach at the middle school or high school level. Table 2 provides a breakdown of the grade-levels each teacher self-reported as their primary teaching responsibility.

Table 2

Number of teachers per grade level

Grade Level	Number of Participants
Early Elementary (K – 2)	1
Late Elementary (3 – 5)	2
Middle (7 – 8)	5
High School (9 – 12)	12
College or University	3

Each teacher's response for each prompt related to each SMP was compiled. The responses were qualitatively examined. An initial researcher using grounded theory principles (Strauss & Corbin, 1998) performed the primary analysis and coding by looking for emerging and cross-cutting themes. For reliability purposes, two additional researchers conducted a secondary analysis of the emerging themes and codes delineated through the initial analysis. Any discrepancies among the three analyses were discussed and reconciled through face-to-face and electronic communications.

Due to the differences in the descriptions provided for each SMP, emerging themes and codes for teacher responses for Prompt 1 were classified with respect to each individual SMP. On the other hand, in examining responses to Prompt 2, although the standards differ, teacher responses were such that emerging themes allowed for categorization by a singular classification scheme. Due to page limitations for papers in these proceedings, we share only the results for both prompts across the first four SMP.

Table 3 presents the classifications that emerged for the first four SMP, and the counts for the number of teachers associated with each classification. If a teacher's response was categorized under two or more classifications, each was counted. Counts were not recorded as to whether or not a response was made only one time, a response was unrelated to the standard, or if no response was made. Consequently, the total count is not always 23 for each standard. For example, of the responses analyzed for Standard 1, 26 themes emerged and were cross-cut, linked, and categorized into the four classifications. This indicates that at least one person responded in a way that allowed for multiple classification of what he or she identified as noteworthy in SMP 1. Alternatively, we were able to only categorize 15 themes among the responses for Standard 4 into two classifications. This lower number of themes for SMP 4 largely indicates a lack of response or responses did not address the standard.

Table 3

Classifications and Counts of Responses for Prompt 1

Standard	Classification	Number of Times Identified
1. Make sense of problems and persevere in solving them	Persevere	7
	Making Sense	8
	Checking Answers	7
	Explaining (Ability to)	4
2. Reason abstractly and quantitatively	(Coherent) Representations	7
	Contextualize/Decontextualize	4
	Abstract Thinking/Reasoning	5
	Meaning of Quantities	7
3. Construct viable arguments and critique the reasoning of others	Critique	3
	Justify Answers/Conclusions	4
	Distinguish Correct and Flawed logic	5
	Listen/Read/Ask	7
	Construct Arguments	3
4. Model with mathematics	Solve Problems in Everyday Life	10
	Assumption, Approximation, Revision	5

Themes in the responses for Prompt 2 emerged as being either student oriented, or teacher oriented. That is, whether the action was something the teacher personally intended to take, or if the teacher's response to the prompt implied action the teacher intended students to take. Responses categorized as student oriented were further analyzed, and themes emerged that either reflected a "student allowance" – a teacher action resulting in something the student would be allowed to do, or reflected a "student need, self-action, responsibility" – a teacher action reflecting student action that is needed in order to meet the a particular SMP.

Similarly, responses categorized as teacher oriented were further analyzed and themes emerged that either reflected "teacher assessment" – a student action the teacher would assess, or reflected "teacher pedagogical/instructional" – a specific action the teacher would take in regard to her or his pedagogical or instructional methods with respect to a particular SMP.

Thus, four categories were used to classify the themes that emerged in the analysis of responses to Prompt 2. Table 4 presents the classification categories, and a sample teacher response to clarify categorizations is provided.

Table 4

Classification Categories for Prompt 2

Classification Category	Sample Teacher Comment per Classification
0. No response or response did not address Prompt 2	Quantitatively is the easy part, thinking abstractly is the harder part.
1. SOA – Student Oriented, Allowances	Allow students to develop reasoning and concepts through problem solving and exploring a variety of contexts.
2. SON – Student Oriented, Need, Self-Action, Student Responsibility	Making sure that students understand symbols and equations in order to be able to read problems and translate into mathematical equations.
3. TOA – Teacher Oriented, Assessment	I will award and/or acknowledge students for partial success rather than all or nothing.
4. TOP – Teacher Oriented, Pedagogical/Instructional	….I will do all steps to the problems out loud and explain why I did the steps and what I was thinking.

Counts for each classification category for the first four SMP are shown in Table 5. Themes in each teacher response were coded into one or more of the categories. In some cases, multiple themes emerged in more lengthy responses that allowed the response to be classified into two or more categories, or alternatively the response did not address the standard or was left blank. Hence the total per standard in Table 5 is not always 23.

Table 5

Counts Per Classification Category for Prompt 2

	Counts Per Classification Category				
Standard for Mathematical Practice	0. No	1. SOA	2. SON	3. TOA	4. TOP
1	0	8	6	2	15
2	1	2	6	3	15
3	1	1	10	1	12
4	0	0	4	1	18

Findings and Implications

Noteworthy aspects of the standards

These data provide answers and insights into Research Question 1, specifically identifying what caught the eyes of teachers as they read the SMP. Although the description of each standard is articulated in relatively few sentences, when teachers read the standards, different items of interest are noticed. All of the responses for each SMP, except for SMP 4, were categorized into at least four themes according to our classification scheme. With only two categories, responses for SMP 4 are more homogeneous and one category contained two-thirds of the responses.

The authors of the standards included key elements in each standard, but teachers seemingly took notice of certain aspects at the expense of other aspects. For example, in SMP 1, "Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends" (p. 6). Only 4 of the 26 (15.3%) teachers' responses to this standard were categorized as *Explaining (ability to)*, the ability to "explain" in this standard specifically pertains to proficiency language. That is, nearly 85% of the teachers' responses seemingly overlooked this explicit language tied to mathematical proficiency.

Intentions to address SMP in instruction

The largest number of counts for each standard was in Category 4: *Teacher Oriented – Pedagogical/Instructional*. The focus on teacher oriented pedagogical and instructional moves is not unexpected. Teachers likely feel most in control of their instructional and pedagogical choices, and as such, likely oriented their reflections towards what they feel they can most control. The second largest number of counts in SMP 2, 3, and 4 was in Category 2: *Student Oriented, Need, Self-Action, Student Responsibility*. Similarly, this is an aspect of teaching over which teachers likely feel they have some direct immediate control. In other words, in identifying aspects of their teaching in which they can be intentional about implementing the SMP, it is not surprising that teachers focus on their practice, and what they perceive students need to be doing to attain proficiency. As such, these data provide answers and insights into Research Question 2. That is, the data identify a certain "status" (i.e., Heck et al., 2011 recommendation) of where teachers are in their thinking on implementing the SMP, and what they must do to be intentional implementers.

Summary

Importantly, little is known about teachers' perceptions when initially reading the SMP with regard to intentionally implementing them in their classrooms. The ideas presented in this paper provide initial baseline data that represents a "status" of teachers as they embark on the implementation of the SMP. The information and data from this research will be helpful to the field so that research on the initial implementation efforts of the SMP does not get lost in the fervor attached to the assessments being constructed to align to the CCSSM.

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KINDERGARTEN STUDENTS EXPLORING BIG IDEAS: AN EVOLUTION FOR TEACHERS

Melfried Olson
University of Hawai‘i
melfried@hawaii.edu

Fay Zenigami
University of Hawai‘i
zenigami@hawaii.edu

This paper describes the experiences of a team of kindergarten teachers as they came to grips with what ideas students could meaningfully explore, represent, and communicate. We share how teachers collaboratively planned a problem-solving lesson for their students and in the process explored the mathematics for themselves as well. We discuss students' work demonstrating their engagement in the mathematics of the lesson and teachers' reflections on the lesson, thoughts about the significance of students' thinking, and efforts to orchestrate communication.

Often kindergarten teachers think they have little influence on the mathematics learning of their children because they do not see the significance of ideas with which children can work. This paper describes a three-year effort made by all kindergarten teachers in one school to understand and influence their students' thinking about mathematical ideas. We discuss a professional development program that had a major influence on how teachers felt they impacted the mathematical development of their students. We explain how a Lesson Study process helped facilitate growth in teacher knowledge as they sought appropriate tasks in which to engage students, expand teachers' efforts to orchestrate opportunities for students to model and represent mathematical ideas, and make visible decisions teachers made to foster discourse to benefit the learning of all students by choosing the order in which students shared their work.

Theoretical Framework

The goal of the three-year project (KARES) funded through a Mathematics and Science Partnerships grant from the Hawai‘i Department of Education was to promote deeper student understanding of mathematics through professional development designed to improve teachers' mathematics content understanding and instructional strategies. The authors were the KARES Project Director and Project Manager, respectively, and were involved in conducting all aspects of the professional development. Carpenter, Blanton, Cobb, Franke, Kaput & McCain (2004) state that the two most critical things teachers need to learn are content knowledge and student learning trajectories specific to that knowledge. Ball (2003) and her colleagues found that when professional development for teachers focused on the mathematical content knowledge they need for teaching, the effects are maximized. Such professional development situated in context is a dynamic approach to learning content that incorporates the idea that content cannot be presented in a way that is divorced from what goes on in the classroom. The objectives of the project included developing deep mathematics content knowledge for teachers; increasing teachers' use of instructional strategies that include problem solving, reasoning, and communication; create and evaluate Educative Curriculum Materials (Davis and Krajcik, 2005) that strengthen teacher and student mathematics content knowledge; and develop a professional teaching community through the use of Lesson Study (Lewis, Perry, & Murata, 2006).

Methodology

KARES was conceptualized in three phases: Phase 1 involved an in-depth look at content of elementary school mathematics. Phase 2 involved researching and investigating selected content at each grade level, with a focus on developing related curriculum materials. Phase 3 involved planning and conducting research lessons to investigate the interaction between teaching and learning at each grade level. The phases were not necessarily disjoint. Because they more directly relate to this paper, Phases 2 and 3 are described relative to the effort of the kindergarten teachers.

Phase 2 - Researching the content at the kindergarten level

The kindergarten teachers began Phase 2 by investigating the Hawai‘i Content and Performance Standards, National Council of Teachers of Mathematics (NCTM) focal points for kindergarten (NCTM, 2008), and initial drafts of the Common Core State Standards for Mathematics (CCSSM) (National Governors Association Center for Best Practices, Council of Chief State School Officers [CCSSI], 2010). They discussed what it meant for kindergarten students to understand the NCTM focal points and how the ideas compared and contrasted with the kindergarten standards in the CCSSM. As they were researching the CCSSM kindergarten standards the teachers engaged in lengthy discussions as to the meaning of some standards and developmentally appropriate instructional sequences needed to implement the intent of the standards. They eventually focused on operations and numbers for their research, materials development, and lesson development. These discussions helped them in their planning for Phase 3. When documenting their planning, Higa-Funada (2011) wrote, “When planning for the first lesson in 2011 – 2012 we examined the CCSD (her acronym for CCSSM). As it was early in the year, we wanted to focus on multiple ways to know a number. Even though the CCSD standard for this was more limiting than we thought, we decided to delve into...ways of knowing 10.”

While thinking about this problem, the teachers engaged in problem solving that expanded their own mathematical knowledge. Even though the CCSSM kindergarten standards state that students “fluently add and subtract within 5” (CCSSI, 2010, p.11), the teachers felt that it would be more robust to explore 10. During their discussion, the question ‘How many ways can we make 10?’ was raised and as a group they engaged in figuring this out. Although several remembered studying ‘something like this’, no one readily knew the solution, but all wanted to know. With some guidance, they explored the patterns found with addition by exploring ‘How many ways can 1 be made?’ (One way: 1), ‘How many ways can 2 be made?’ (Two ways: 1+1, 2), ‘How many ways can 3 be made?’ (Four ways: 1+1+1, 1+2, 2+1, 3), etc., and were pleased with their problem solving abilities to arrive at a solution. They also realized that the large number of possible ways might make the problem *more accessible* for their students.

Phase 3 - Developing and planning a research lesson

Phase 3 was designed so teachers would prepare lessons, research how the lessons worked with children, and reflect on the experience. Teachers engaged in the typical Lesson Study process involving (1) defining an instructional problem, (2) researching the problem and brainstorming possible lessons that could speak to the issues, (3) anticipating possible student misconceptions that might occur, and (4) then, planning the lesson. While step (2) is often viewed as the most crucial, all steps should occur before the lesson is taught. This was important because the research and deep thinking the teachers did before the teaching gave them ownership of a lesson they planned and prepared to teach. In planning the research lesson, teachers used the research evidence they collected to prepare the lesson. When preparing the lesson they used a format with three columns (Figure 1). In the first column they listed the planned sequence of events that would occur as the lesson unfolded, including key questions to ask and the rationale for asking them. In the second column, they listed reactions and responses they anticipated students would make. These responses were based on evidence they found in research as well as from their prior teaching experiences. Column 3 had suggestions for how the teacher could respond to the anticipated responses. The third column was also used to record student responses and to think about how to use these responses when conducting whole class discussion.

<table>
<tr><td colspan="3">Kindergarten Lesson Format

How May Ways Can you Make 10?</td></tr>
<tr><td>Steps of the lesson: Learning activities and key questions with rationale (time allocation)</td><td>Anticipated student reactions or responses</td><td>Teacher responses to student reactions. Things to remember.</td></tr>
</table>

Figure 1. Format used for preparing lesson

Studying the Lesson

To maximize the opportunity to study and reflect on the lesson while at the same time causing the least disruption for the teachers and students, the following process was used: one teacher taught the lesson while all others observed; this teacher led a debriefing session that occurred immediately after the lesson was taught; if deemed necessary, minor suggestions for adjusting the lesson were made; a second teacher taught the same lesson; and that second teacher led the next debriefing session. While not a part of the study sequence involving observers and debriefing, the other three kindergarten teachers later taught the same research lesson.

The goal of the research lesson, How many ways can you make 10?, was to develop a child's sense for "ten-ness" which the teachers knew would support later work with numbers between 10 and 20. As they brainstormed how to provide a context for their students, they realized they had a significant amount of scrip remaining from the school's family fair. They created a story for the students about how a collection of 10 scrip could be exchanged for an "ice pop" treat left over from the fair, and prepared curriculum materials to deliver their lesson.

For the lesson, students were provided containers of scrip with multiple lengths of 1, 2, 3, or 4 scrip per container and asked to make a collection of 10 using any combination of scrip they wanted. When students were sure they had 10 scrip, they were to glue the scrip on a strip of construction paper. Students were given enough time to make one set of 10 and encouraged to make more collections if they could. This task was accessible to all students, and the variety of responses was interesting, but more interesting was the discussion in which the teachers engaged the children. Samples of student work are in Figures 2 – 4.

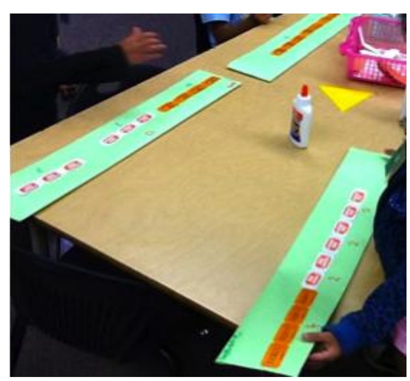

Figure 2

In Figure 2, both students worked with the largest scrip piece (4), but one used two 3s while the other used three 2s. This aspect later proved interesting as when asked to count to verify totals, some students used a skip counting process rather than counting by 1s. In Figure 3, a student used three 3s and a 1 to make 10. Several students used a similar configuration, but in a different order. This provided an opportunity for the teacher to compare and contrast how

Figure 3

Figure 4

arrangements were similar or different. Figure 4 shows a student had completed two combinations of 10 and was working on a third.

After allowing time for each child to make a collection of 10, the teacher called the class to sit on the carpet at the front of the room with their work. Individual students were asked to show their collection and tell the numbers involved. For example, the student who showed the collection in Figure 3 would say, “3, 3, 1, 3.” As was planned in the lesson, the teacher at this point, while holding two different collections, engaged students in exploring, “Is 3, 1, 3, 3 the same or different from 3, 3, 1, 3?” Teachers were excited by the appropriate students’ responses.

On a day following the research lesson, one teacher worked with a group of students to further talk about similarities and differences. She had students share an example they had made or could make and recorded their suggestions on poster paper as shown in Figure 5. She made the effort to both draw the arrangement and to use the numerical values in the discussion and had students describe a collection, such as Jaycha’s, by saying, “One scrip and two scrip and two scrip and two scrip and three scrip make a collection with 10 scrip.”

Figure 5

Teacher reflection on the Lesson Study Process

After all five teachers had taught the lesson, they responded to the following prompt, “How successful do you think the problem your team selected for the lesson study addressed your team's goals? As much as possible, cite specific examples from your observations." The collective response of the kindergarten teachers is given in two parts below. One part pertains to how the lesson matched their intended goal, and a second part pertains to what the teachers felt they learned from the work of the children during the lesson.

***Response related to how the lesson matched the intended goal*:**

> The K team's goal was to develop number sense and experience making groups or combinations of 10. The problem we chose addressed a wide range of academic abilities (we were able to differentiate between levels). Examples are:
> 1. Some children from various classes took the initiative to add the sets of 10 together.
> 2. We were also able to identify students who need more support and practice with one-to-one correspondence.
> 3. Even the ones which were wrong (were) used as teachable moments in which the students were taking the lead as problem solvers.
> 4. Students developed oral communication when sharing their combination of 10. Even the less verbal students felt confident in sharing.
> 5. Problem was relevant and had a personal connection to their own experiences using scrip at the Kapālama Family Fair/Chuck E Chesse/Fun Factory. (Sakumoto, 2011).

***Responses related to what was learned from student responses*:**

> The Kinderettes (Author Note: What they called themselves.) felt surprised that the problem we chose exceeded our expectations in the following ways. Children were able to: self correct/identify how to help others; collaborate with each other; show many combinations of 10, compare and contrast their own combinations with their peers; subitize 1-4 scrip (Author Note: This is something they learned about in their research.); see the big idea that there are a variety of ways to make 10; verbalize their thinking; and take the initiative to extend the lesson. (Author Note: One example of this initiative occurred when students had more than one scrip of the same length, such as 2 + 2 + 2 + 2 + 2, and saw they could skip count to validate this was a collection of 10. A second extension occurred when one child suggested they count the total of all the collections made. While the student did not count by 10 to determine his solution, his final total was close to the correct value.) (Sakumoto, 2011).

Findings, Discussion, and Conclusions

The teachers and authors made observations related to several aspects about the research lesson that was conducted. These comments demonstrate that teachers made advances in the way they thought about the mathematics they were teaching, the lessons they were preparing, and in their analysis of student learning. The comments, with supporting evidence, are listed below:

1. Teachers designed an appropriate lesson. Each time the lesson was taught, the children were actively engaged in the problem and intent on learning. Students were on task and most

found multiple correct solutions. This is a tribute to the depth of thinking the teachers did in deciding on the task to investigate, and in knowing the significance of the mathematics involved.

2. The effort teachers made to fine tune the task paid off. The task was challenging but accessible. While the CCSSM only focuses on knowing numbers 1 – 5, each child was able to work successfully on this task. Even those students who basically could only approach the problem by putting tickets one at a time were able to engage meaningfully with the task.

3. The teachers were able to use discourse and communication in a meaningful way. Because of the specific structure incorporated into the lesson, with anticipated responses and suggestions for handling the responses, communication options occurred they never considered before. Students who struggled to find a solution still were comfortable sharing what they made and why they thought they made 10. Similarly, when a student had an incorrect solution, he or she could count and determine that it was not 10. When either of these things happened, teachers asked other students, "How can we help _____ make 10?" Interestingly, students were able to provide multiple answers to this question. This shows the benefits of effective planning for student responses and actions the teacher should be prepared to take.

4. The task proved well suited for the English Language Learners students in the class. They could do the mathematics expected of them while being coached through the words needed to explain their work. The decision the teachers made to have students record the value of each length of scrip used when making their collections provided a representation that helped prepare the students for the explanations they were to give. As a result of their research, teachers became more aware that a focus on communication in mathematics supports other parts of the curriculum.

5. Interesting discussions were forthcoming when the teachers asked, "How are these the same, and how are they different?" While children realized they were all showing 10, the discussion of same and different led to excellent comparisons. Some children said a 2, 3, 3, 2 and a 2, 2, 3, 3 were the same because they each used two 2s and two 3s, while others said they were different because the order was different. Some saw that 1, 2, 3, 2, 2 and 2, 2, 3, 2, 1 were 'reversed'. The language used in the discussions exceeded teachers' expectations, suggesting that the preliminary discussions the teachers had about these same questions filtered into their lessons and instruction.

6. While preparing the lesson, teachers became focused on how to use a context to which their children could relate. This led them to focus on the development of the scrip and the “ice pop” story. We all observed that throughout the lesson the students were so engaged in the thinking and reasoning involved that the story had become irrelevant.

When reflecting on how focused the student were on this lesson and they forgot the story the teachers used to get started, one teacher summed up the positive aspects of their effort to create this lesson when she said, “It was not about the ice pops, it was about the learning.” This statement captured the essence of the evolutionary journey to involve kindergarten students in the exploration of big mathematical ideas.

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THE IMPACT OF LONG-TERM PROFESSIONAL DEVELOPMENT ON TEACHER SELF-EFFICACY: A CASE STUDY

Eileen Durand Faulkenberry
Texas A&M University-Commerce
Eileen.Faulkenberry@tamuc.edu

Maribeth Nottingham
Southeastern Oklahoma State University
mnottingham@se.edu

This study examined the impact of a two-year professional development on self-efficacy of a teacher participant and the effects it had on teacher practice and student outcomes. The professional development was focused on instructional practices to enhance student learning; was based on national and state standards; and incorporated peer collaboration with classroom observations and support. Qualitative assessments of the participants showed growth. This study includes an in-depth case study of one of the participants showcasing her significant growth in confidence and improvements in her instructional practices.

Effective professional development builds on teacher's prior knowledge and engages the teachers in learning that connects to their personal jobs, sense of community/collaborations and sustains long-term instructions (Hunzicker, 2010; Buczynski & Hansen, 2010). When the educator is able to incorporate the training into practice, making connections and given opportunities to take this training experience back into the classroom, then the professional development translates itself into something useful, memorable and becomes effective teaching. Authentic and supported development must be "job-embedded" and connected to the daily routine of the teacher (Hunzicker, 2010). This form of professional development allows the teacher to take risks, try new learning methods and strategies, and learn how to analyze their teaching style and student outcomes for effective lessons.

Self-Efficacy is dependent upon a teacher having a good working content knowledge, an understanding of how students learn, and good motivational, reflective skills and methods (Bandura, 1993). Researchers Cantrell and Hughes (2008), in their study, *Teacher Efficacy and Content Literacy implementation: An Exploration of the Effects of Extended Professional Development with Coaching*, found professional development may promote increased implementation and efficacy.

The purpose of this project was to provide a two-year professional development through a Math and Science Teacher Academy (MASTA) grant project. The professional development features the critical components detailed by Desimone (2009). It is a long-term, supported professional development that is focused on content in mathematics and science, includes active learning and collective participation using collaborative learning, and supports the growth in teacher participants' knowledge and beliefs. The project consisted of two-week summer institutes that focused on intensive content and pedagogical content knowledge (Shulman, 1986; Hill, Ball, Schilling, 2008). We then met twice monthly, once face-to-face at the university and once online. In the face-to-face sessions, we presented math and science content using strategies to promote student learning, modeling classroom practices for the participants to take back to their classrooms. The online meetings focused on the judicial use of technology and best practices for incorporating technology to support learning. This professional development for mathematics and science teachers provided inquiry-based methods including Japanese Lesson Study, Problem-Based Learning, and Action Research. Research based instructional practices were modeled with the intent of teachers incorporating these strategies into their classrooms. The professional development also focused on the judicious use of technology in the classroom to enhance student learning. Data collected for the professional development project allows the researchers to explore the effects of a long-term professional development with focused math and science instruction on teacher perceptions, classroom practices, and student learning.

Inquiry-based learning methods were chosen as the basis for the professional development for their student driven strategies, higher thinking objectives and constructivist teaching philosophies (Curcio, 2002). Inquiry-based learning is a student driven method incorporating collaboration, critical thinking, and instructionally focused teaching/learning methodology connecting prior leaning with present learning. Inquiry-based learning is a process of answering questions through learning, not only using acquired knowledge in content areas, but using prior knowledge to make connections.

Data collected throughout this project included classroom observations, interviews, surveys, and classroom level student achievement reports. In order to measure changes in teacher self-efficacy, the Teacher Self-Efficacy Scale (Woolfolk and Hoy, 1998) was used. The scale consists of forty-three items across three subscales in Likert format. The Likert scale was designed to measure opinions and attitudes on a scale scoring from "strongly agree" to "strongly disagree," each being given a number score with values from one to nine, with nine representing "strongly agree." Mean scores were used to determine high and low areas of teacher attitudes and beliefs with regard to their knowledge of content and pedagogy. This scale was designed to elicit the teachers' beliefs based on their actions in the classroom and how the actions impact student learning as well as the factors outside of the teachers' control. This same scale was given within thirty days of the very first meeting in 2009 and again in the summer of 2011 at the close of the summer course. The teacher efficacy scale will help to build a picture of efficacy growth or lack of growth during the two-year professional development time span.

Classroom observations used the Reformed Teaching Observation Protocol (RTOP) (MacIsaac, Sawada, Falconer, 2001). The RTOP gives the researcher the opportunity to take a comprehensive snapshot of the classroom learning environment. The protocol has both qualitative and quantitative sections regarding classroom demographics, interactions, content and implementation of lessons. Classroom observations were conducted during the spring semester of the first year and throughout the second year.

There were 53 participants in the professional development project from 17 public school districts and two private schools. The average number of years teaching was 8 years with a range of 0 to 24 years. There were 45 females and 8 males. The participants taught math and/or science in grades 3 through 12.

Mrs. A had been teaching 2 years prior to the beginning of the project. She was certified through an alternative certification program. She had been an accountant prior to coming to teaching. She has fewer than 12 hours of college level mathematics courses, but she has passed current state licensing exams. She taught 7th grade math in a rural school district with 53% of the students classified as low socioeconomic status. 48% of the students in the district are white, 27% hispanic, and 25% African American. Mrs. A had an average of 73 students each year.

Mrs. A reported that she applied for the professional development project because she wanted to "learn more ways to be a better teacher." Initial classroom observations showed that Mrs. A exhbited a tentativeness in her lessons. She seemed to lack confidence in her content knowledge as well as her ability to manage the learning environment in her classroom. The students were working in groups, but Mrs. A did not facilitate the group learning effectively to ensure that all students were on task and learning the material. In initial surveys regarding technology use in the classroom, Mrs. A responded that she had "no talent whatsoever" with technology but that she really wanted to use technology in her classroom. "I would like to know how to use more technology. I would like know how to use calculators more effectively as a learning tool. I have [an interactive] board but I don't know what to do with it." On the initial teacher self-efficacy scale, Mrs. A had a mean of 3.1 showing that she felt she had some influence on student learning. Her responses showed that she felt her training did not prepare her to help students who had difficulty learning the material. She did not know how to help a student who didn't remember information given in a previous lesson or how to redirect students who were off task.

The professional development had three overarching themes throughout the project: reflective practice, questioning techniques to guide student learning, and the judicious use of technology. These themes influenced all curriculum of the project. Each phase of the project focused on a different method of reflective teaching and learning. The first module included Japanese Lesson Study, a model of professional development that involves collaborative planning, teaching and observing, reflection and revision, and re-teaching in order to create exemplary lessons (Curcio, 2002). This reflective process allows the participants to focus student learning and tailor their classroom practices to enhance student thinking.

The second module focused on Problem Based Learning (PBL), a student centered approach whereby students learn about a given topic through asking questions about and solving realistic problems. The goals of PBL are to help the students develop a flexible knowledge and the ability to solve novel problems. In PBL, students work in groups to solve the given problem. The role of the teacher is that of facilitator, guiding the students through the problem solving process.

The third module used Action Research to focus the lens on how teacher actions and classroom practices impact student learning. In action research, the teacher identifies a question he or she would like to answer regarding classroom practice or student learning. The teacher then collects data to answer the question, analyzes the data, and interprets the results. Llewellyn and VanZee (2010) found that action research in a classroom setting improved confidence in teaching abilities and methods while closing gaps between theory and practice.

After participating in the two year professional development, Mrs. A showed much more confidence in her abilities to guide her students' learning, manage her classroom, and incorporate technology in her lessons. She reported that she had "learned new strategies and technology that I frequently use in my classroom thanks to [this project.] I feel that I am a better teacher now because I am able to offer more variety to my students now." She incorporated technology such as calculators, her interactive white board, and video camera into her lessons. "I would say that I have grown the most in my ability to use technology. Using technology in various ways keeps students engaged and wanting to learn. Without that interest, how can they learn anything? I used the flip camera to video students. I would then play the clips for students to defend or constructively criticize. This improved the students' ability to communicate mathematically and support their thinking." Classroom observations showed that she had seamlessly integrated the use of the interactive whiteboard into her classroom practices.

On the final teacher self-efficacy scale, Mrs. A had a mean of 7.2, up from 3.1 initially. She showed growth in the ability to craft good questions for her students, use a variety of assessment strategies, and provide alternative explanations when students are confused. She also felt more secure in her ability to manage her classroom, stating that she felt more comfortable in her ability to control disruptive behavior in the classroom, establish routines to keep activities running smoothly, and calm a disruptive student.

Classroom observations showed that Mrs. A's lessons were trending toward more student centered lessons, effectively using group work, facilitating the learning through scaffolding questions while the students were solving novel problems. Her lessons promoted coherent conceptual understanding, multiple representations and mathematical communication of strategies and solutions. Her students showed great respect for classmates as they were sharing their thinking.

Mrs. A reflected on the action research portion of the project with the following statement.

> "I was anxious and nervous about the action research project when I first learned about it last fall. I didn't know how to display the information, what information I needed to collect or even my 'burning question?' In fact, I did not know what a burning question was at that time. I brainstormed several things that I have always wondered about. Would this help in my classroom? Would that be advantageous for my students? Some of my ideas were suitable for research and some were not. I finally decided to try singing math songs that relate to the lesson in my classroom.

I had never been brave enough to try this before in my classes because I am definitely not the next 'American Idol.' My research showed a ten percent increase on test scores on lessons accompanied by a math song. I found this information valuable and plan to integrate more music in the future. I also plan to use more action research in my classes to answer questions I have."

Interviews conducted with Mrs. A at the end of the project showed that she differentiates her teaching style dependent upon the content of the lesson.

"My stategies often depend on the concept. There are some that I will explain through direct instruction and some I prefer for the students to explore and find on their own through student centered discovery learning. I couldn't find a good hands-on activity to model division of fraction so I used direct instruction with the smartboard and lots questions. For our circumference of a circle lesson, the lesson was student centered and the discovered the relationship of the diameter and pi through their own exploration." She stated that "I ask 'why' often. It is easier to give the correct answer than it is to say why it is correct. A deeper understanding of the concept is necessary rather than just going through the algorithms and mindlessly following steps. 'What if" is also a good start of a higher level thinking question. I demand participation in classroom discussions and everyone contributes. I often have problems where student must justify their answer using a written description that they then share with the class. We also use a variety of representations from graphs and graphic organizers to color tiles and centimeter cubes. I love to use as many hands-on activities as possible because I can tell the students get so much more out of it!"

State assessment data showed that over 90% of Mrs. A's students met or exceeded the standards set by state, a level she had not previously reached. Almost one-third of her students were at the commended level. Her students' results were significantly different from those of the rest of the school.

This study examined the effects of long term professional development on teacher self-efficacy and classroom practice. Through a study of Mrs. A, we found that she felt more confident in her abilities to manage her classroom and facilitate student learning as a result of the professional development. She is using more research based instructional practices in her classroom. She has successfully incorporated technology into her lessons. She has focused on student thinking and arranges her instruction based on the needs of her students. Her lessons are becoming more student centered, encouraging her students to solve novel problems and share their solutions. Through this project, she has become a more reflective teacher, striving to improve her teaching through action research and collaboration with colleagues.

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THE PATH OF REFORM IN SECONDARY MATHEMATICS CLASSROOMS: SOME ISSUES AND SOME HOPE

Michael Mikusa
Kent State University
mmikusa@kent.edu

Joanne Caniglia
Kent State University
jcanigl1@kent.edu

Scott Courtney
Kent State University
scourtn5@kent.edu

In this study an entire mathematics faculty (11 secondary mathematics teachers and 4 intervention specialists) were engaged in professional development over 2 years. The professional development activities were aimed at improving the learning of mathematics by all students at the school. The PD ranged from mathematics content for the teachers where they engaged in solving rich problems to assessing video tapes of teachers teaching and students learning mathematics from a constructivist perspective. This paper focuses on some of the key issues that kept these teachers from reforming their classroom consistent with what they were learning during the professional development.

The TIMSS data indicates that our students are not keeping up with the rest of the world in their performance in mathematics. A video study that was completed as part of the TIMSS project indicates that teaching in secondary schools is also in need of professional development. In their study, Hiebert and Stigler (2004) describe the shortcomings of current secondary teaching in the video case studies from the 1999 TIMSS study. "Although teachers in the United States presented problems of both types (practicing skills vs. 'making connections'), they did something different than their international colleagues when working on the conceptual problems with students. For these problems, they almost always stepped in and did the work for the students or ignored the conceptual aspect of the problem when discussing it." One reason for this type of behavior may be that secondary teachers need experience themselves in solving rich problems and building connections between and among the different topics within mathematics before they can successfully help students be more engaged in this type of teaching and learning.

The Mathematical Association of America (MAA), comprised of mathematicians and mathematics educators, explains in its landmark document, The Mathematical Education of Teachers (2001), that the mathematical knowledge needed for teaching is quite different from that required by persons in other mathematics-related professions. Teachers need an especially profound understanding of the concepts of mathematics so that they can teach it as a coherent, sense-making, reasoned activity. For this to be accomplished, MAA recommends collaboration between mathematicians and mathematics educators with close connections to classroom practice. The MET document further suggests that professional development should be aimed at: deepening the understanding of the fundamental mathematics beyond what is usually reached with an integration of content knowledge, cognitive science, and teaching experience; to develop effective "habits of mind" for thinking mathematically; to develop realistic strategies for in-depth implementation of the NCTM Process Standards; to analyze teaching practices in light of deep content and mathematical pedagogical knowledge; to reflect upon personal beliefs about mathematics content and pedagogy over time; to develop a high level of confidence and a positive disposition for mathematics in order to provide appropriate instruction for all children and gain parental support and understanding (CBMS, 2001). In order for the practice of secondary teachers to change with the recommendations in the MET document teachers must themselves engage in looking at the mathematics they are teaching from different points of view. They need to develop a more comprehensive understanding of the connections of the mathematical ideas being taught and develop a collaborative network of support for their work on the content and on implementing various changes within their classrooms. Improving achievement in mathematics education involves many components. Current research focuses on strengthening content knowledge of teachers that will enable them to understand mathematical connections among ideas and the sequence of ideas that make sense both mathematically and developmentally (Ferrini-Mundy, & Schmidt, 2007, Ball, Thames, & Phelps, 2008).

This paper reports the results of a study of the effects of a professional development partnership with an entire mathematics department and 4 intervention specialists in an urban school district in NE Ohio, three mathematics educators and three mathematicians from a local university. The goals of the professional development were to: 1) Deepen teacher mathematical content knowledge and knowledge for teaching; 2) Create and maintain a professional learning community; 3) Transform teachers' ways of thinking to incorporate dispositions that hold students accountable for engaging in the common core standards for mathematical practice; and 4) Project sustainability through the creation of a community of vested partners, where partners interpret, enact, and sustain the Common Core Standards vision of developing mathematically competent students. Ultimately, the aim was to improve the teaching and learning of mathematics at the high school level.

Our collaboration model was formed using a combination of existing models: the Yale/New Haven National Teacher Institute Model and the Focus on Mathematics (Boston University/Educational Development Center's National Science Foundation's MSP). Two premises were followed from these models: all partners involved had equal say in the direction of the professional development and that there would be core involvement of mathematicians in the partnership. The high school teachers and university faculty used the hybrid model to design the professional development opportunities that were cogenerated. This co-planning of the direction of the professional development was enacted in face-to-face meetings each week of the school year as well as 2 weeks in late summer and 2 days after each school year was completed. Ideas from the project staff were presented and specific teacher problems and issues were presented by the participants. Discussions took place to determine the most needed professional development.

Methodology

Participants

All eleven mathematics faculty and 4 intervention specialists were part of this study. All of the teachers have the appropriate license for their teaching assignments. The teachers ranged in experience from 2 years to 25 years. All signed an agreement to participate in the grant and in return the grant provide the professional development, $500 in teaching materials for their classrooms, and stipends when PD occurred outside their regular teaching contracts.

The treatment/professional development

Each year the teachers had approximately 100 hours of professional development focused at the goals mentioned above. This occurred during summers (60 hours), each week prior to classes starting (20 hours), and once a month (20 hours) during an early release time.

During the summers teachers were engaged in doing mathematics problems. This is where our mathematicians helped to lead these discussions about the problems, various methods of thinking about the problems, reasoning for solutions, and connections to other areas of mathematics and the common core state standards. During this time, one goal was to build confidence and community among all participants. We collected samples of the rich problems teachers worked and videotaped these experiences. We also surveyed the teachers on their beliefs about the nature of mathematics and the teaching and learning of mathematics.

During the academic year teachers were grouped with common courses and together prepared a common lesson. A mathematics topic was discussed, lesson goals were developed, a task was identified and each teacher in the group taught the common lesson to one of their classes while the others in the group observed both teacher and students. The teachers in this group met prior to school starting to coordinate the day, and immediately after each teacher in the group taught the lesson to debrief about what we observed. The final debriefing involved some analysis and recommendations for future teaching. These common lessons yielded one common comment by all these teachers; "…we need to have the students do more of the work". This common finding is in agreement with what Hiebert and Stigler (2004) found in their video analysis. Because of these experiences the teachers recognized and vocalized the need to get their students more involved in the lessons and have them take more responsibility for their own learning of mathematics.

Teachers and university partners also met regularly to work with the group on common assessments (a school-wide initiative) and adjustments to their curriculum. There were also many small group-based requests for non-regular meetings. An example of these non-regular meetings was the planning of a family mathematics night where teachers took charge of preparing demonstrations for students and parents of the changes to the curriculum and teaching. In addition, some teachers wanted to know more about differences in learning of students from urban areas and low socio-economic backgrounds. We formed a reading group for the book Teaching with Poverty in Mind: What Being Poor Does to Kids' Brains and What Schools Can Do about it by Eric Jensen. (2009, ASCD, Alexandria, VA). Additional activities with small groups of teachers (never single teachers working) were supported though out the three years. Each of the professional development activities helped to form stronger knowledge of the content, how students learned the content, or to build a professional learning community among all the partners involved in the grant.

Instruments and measures

A mixed methods design for this research was implemented to measure a complex set of variables that may affect the teaching and learning of mathematics. Data of the content knowledge of the teachers was gathered using an adaptation (for high school teachers) of the LMT (Learning of Mathematics for Teaching) assessment and a collection of artifacts (solutions to the rich problem tasks, observations of teachers using the RTOP, and interviews by the internal evaluator) were analyzed. We also administered a survey of the nature of mathematics and mathematics teaching and learning to the teachers. The adaptation of the LMT consisted of choosing only those items that pertained to upper elementary mathematics topics. Topics of ratio and proportion, geometry and reasoning, and fundamental concepts of algebra were included. The test had 11 separate items, a few with 3 or 4 parts, and all items were multiple choice. The modified LMT given toward the beginning of the PD work and at the end of year one had similar items, however, because of these modifications the scores were used to identify where at least half of the faculty involved had misconceptions. The analysis of the rich problems also helped to characterize the knowledge that these mathematics teachers displayed in PD activities.

Results

One of the problems, as is a trait of the LMT, had teachers analyze an alternate method for dividing two fractions. The student said that he divided the two numerators and the two denominators in $\frac{6}{8} \div \frac{1}{2} = \frac{6}{4}$. The teachers were asked to choose one of the following about what this student's teacher was thinking about the students method (number of teachers selecting an answer): a) He knew that the method was wrong, even though he happened to get the right answer for this problem (3); b) He knew that the student's answer was actually wrong (0); c) He knew that the student's answer was right, but that for many numbers this would produce a messy answer (4); d) He knew that the student's method only works for some fractions (6). The correct answer was selected by only 4 of the 13 teachers. Answers **a** and **d** were selected by 9 of these high school mathematics teachers. These two selections imply that the teachers themselves knew that the student had gotten the right answer and that either they didn't or couldn't determine if the work presented was equivalent to the standard algorithm for dividing two fractions. Based on other mathematics problems similar to this question it was clear that these teachers were not used to thinking about alternative methods for solving problems and why various algorithms that they used regularly worked or how they were derived. On an exit interview with the internal evaluator near the end of year 2 all of the teachers except one indicated that they believed their knowledge of mathematics and of the common core state standards had increased. When asked why they believe this to be true they referred to the rich mathematical tasks that they solved and had to demonstrate their reasoning for their solutions. They also indicated that they were more confident in being able to implement the common core state standards as a result of their participation in the grant activities as was also supported by our observations and analysis of the summer professional development activities. We believe this to be a direct result of our efforts at introducing problems that could be solved using both conventional and non-conventional techniques and having to argue to justify whether something worked always, sometimes, or never.

In the analysis of the observation notes and videos of the teachers doing rich problems it was noted that in the early stages of the grant most of the teachers worked the problems alone at first. They only shared out when the faculty asked them to do so. This sharing initially consisted of them showing each other how they did the problems. The more confident the teacher the more they wanted to share. As the grant progressed and teachers had multiple opportunities to solve problems and experience non-conventional approaches, all teachers were more likely to share their methods of solving the problem. We also observed that the teachers began to engage in both presenting their reasoning and listening to and critiquing the reasoning of others. So both our notes and analysis of videos (of teachers engaged in doing rich tasks in the PD) supported that teachers did learn mathematics that would help to support their teaching of high school mathematics.

Table 1 below contains the results of a survey of the teacher's view of the nature of mathematics. They completed this survey after two years of professional development had been completed.

Table 1

MT = Mathematics teacher. Int. Sp. = Intervention Specialist

Subgroup	Part I (Absolutist-Fallibilist)	Part II (Authoritarian-Social Constructivist)
AC – Int. Sp.	3	3.4
JE –MT	3.88	4.4
AB – MT	3.11	3.8
DM – MT	3.66	3
KT - MT	3.33	4.8
MN - MT	3	3.2
TB - MT	3.44	3.4
TR - MT	3.22	4
CW – Int. Sp.	3.22	3.8
SS – Int. Sp.	3.33	4.2
JF - MT	3.33	3

Discussion

On part I of the survey the teacher's chose their view of the nature of mathematics. If the average score was 4 – 5, then the person had a more applied (fallibilist) view of the nature of mathematics. If the average score was 3, then the person had a mixed view of the nature of mathematics. If the average score was 1 – 2, then the person had a more pure (absolutist) view of the nature of mathematics. From the teacher's view of mathematics we see two of them are classified as having a mixed view, while all of the others scored between 3 and 4. While not "clearly" in the Fallibilist camp these teachers did appear to be more open toward the belief that the nature of mathematics may not be fixed and that what is important regarding mathematics is evolving. It is also fair to say that their original belief about mathematics is still nearly as influential despite hundreds of hours of professional development working to open up their views and beliefs.

In part two of the survey the teachers indicated their view of mathematics teaching and learning. If the average score was 4 – 5, then the person had a more social constructivist view of the nature of mathematics education. If the average score was 3, then the person had a mixed view of the nature of mathematics education. If the average score was 1 – 2, then the person had a more authoritarian view of the nature of mathematics education. Four of the teachers scored as being social constructivist, two averaged 3, which means they have a mixed view, and the remaining five teachers scored between 3 and 4. So nine of the teachers seem to believe that the nature of teaching and learning of mathematics is aligned more closely to a social constructivist point of view. We believe that these scores should be interpreted as steps toward changes in teaching. When we reviewed the RTOP scores the teachers were still predominately teaching by telling as opposed to helping students to make sense of the mathematics on their own. This passage from the final report about the RTOP results clarifies how teachers are talking about transforming their classes but they haven't yet made these changes in their own teaching.

> The teaching style is almost exclusively teacher-focused where the teacher maintains the role of the mathematical authority. Teachers have voiced their lack confidence in the students' ability to be challenged either believing the students lack pre-requisite knowledge or/and a disposition toward mathematical thinking and perseverance. This belief plays out in many of the observed lessons. In grant

meetings teachers do voice a desire to move towards students “doing more of the work”. Mirroring some grant activities, there is evidence teachers are including more instructional strategies to move the classroom culture towards student centered. But what that entails can differ according their notation of how mathematical thinking is defined. They tend not to be lecture-based and are looking for ways to engage students. What is missing is allowing students the opportunity to interact with the content in an authentic way (Graham, 2011).

Conclusions

Secondary Mathematics teachers face a multitude of factors in how they think about, prepare, instruct, and assess their students’ mathematics. Some of these factors evolved directly from instruments used to collect data during the grant (Survey, RTOP, and content exam) and some from discussions that occurred during the many PD meetings we had throughout the first two years of the grant. If these factors came from discussions, they were noted only if the factor was brought up by several different members of the faculty and at different times. That is, a discussion issue was the concern of most of the faculty and mentioned at several different professional development meetings. The primary factors that we identified were (and in no particular order of importance): teacher’s own personal beliefs about the nature of mathematics (Survey and PD discussions), how they learned mathematics (Content Exam, PD discussions), the perception of mathematics by administrators (PD discussions), the perception of mathematics of their students (and parents of their students) (RTOP, PD discussions), and probably least influenced by their interactions with each other (PD discussions). For this group of teachers, experiencing three changes in superintendent, three different principals, and two different curriculum directors in the district, it is amazing they were able to maintain any focus on teaching and learning mathematics. To their credit they have been active in our professional development and in their desire to go beyond learning about the changes being proposed from research in mathematics education and the common core state standards. The next step for these teachers is going beyond their initial attempts to implement these changes and beginning to study how what they do makes a difference in the learning of their students. The next phase of our work with these teachers is to involve them in a videotape study of their own teaching with support from colleagues. These classroom videotapes will provide the direction and evidence to help these teachers transform their teaching. We have mentioned many reasons that inhibit a whole department of secondary mathematics teachers in making the shift toward implementing reforms suggested by current research. We are encouraged however with the willingness to work together on the video project phase of our collaboration to see how these changes can be implemented in their own classrooms.

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HAPPILY EVER AFTER: EXAMINING INSERVICE TEACHERS' BELIEFS ABOUT USING CHILDREN'S LITERATURE TO TEACH MATHEMATICS

Ann Wheeler
Texas Woman's University
Awheeler2@twu.edu

In this study, the researcher qualitatively investigated how incorporating children's literature into the mathematics classroom impacted 18 middle school (Grades 6-8) inservice teachers' mathematics pedagogy. Data collection consisted of tests, background surveys, journal article reviews, and daily reflections. Based on data analysis, findings suggest that all participants were open to the idea of using children's fiction in their mathematics classroom with seven claiming to want to incorporate fiction more than ever before in the upcoming years. Implications for college educators include providing teachers with opportunities to see the utilization of children's literature in mathematics lessons.

Oftentimes, K-12 teachers may struggle to capture students' interests in mathematics, where students may feel mathematics is too difficult and/or uninteresting to invest their time and effort. Even though teachers may face opposition from students in learning mathematics, using children's literature in any level of a mathematics classroom can help students improve critical thinking skills, thwart mathematics apprehension, and engage in meaningful mathematics learning through contexts (Furner, Yahya, & Duffy, 2005). In addition, one cannot downplay the valuable mathematically rich exchanges between students and teachers that books afford (Roth-McDuffie & Young, 2003; Thiessen, 2004) or the connections teachers can make between books and mathematically rich games (Cutler, Gilkerson, Parrott, & Bowne, 2003).

Through a grant funded 11-day graduate course, the researcher attempted to qualitatively answer the following overarching research question:

How does children's literature impact inservice teachers' opinions about teaching mathematics?

Literature Review

There is much research that suggests the benefits of children's literature in the mathematics classroom (Cutler, Gilkerson, Parrott, & Bowne, 2003; Furner, Yahya, & Duffy, 2005; Roth-McDuffie & Young, 2003; Thiessen, 2004; Whitin & Whitin, 2004). One of the most obvious examples is via the Communication Standard in NCTM's *Principles and Standards* (2000). NCTM advocates the use of communication in the mathematics classroom, which can be achieved through dialogues between the teacher and students when children's books enter the discussion. Many texts include wild scenarios, such as knights and perilous geometry-inspired quests (Neuschwander, 1999), shape shifting polygons (Burns, 1994), and even dueling mathematically savvy barbers (Sundby, 2000), which promote thought-provoking exchanges of mathematical ideas in ways that formulas and lectures could never do.

Although many authors claim their texts can be used in mathematics classrooms, caution should be used when making your choices. Whitin and Whitin (2004) believe there are four main areas to consider when selecting a book. The first main concern is that the mathematics being portrayed in the story is sound and authentic. Stories should not be manufactured as a way to sell books. In addition, the book’s tone should elicit lively discussion. Thirdly, the words and/or representations in the book, such as pictures and graphics, should provoke the reader’s attention. Lastly, Whitin and Whitin detail the importance of social justice issues, where race, gender, and culture are all respectively addressed by the author(s) and illustrator(s).

Methodology

For this study, the researcher focused on middle school (Grades 6-8) inservice teachers who agreed to be a part of a one-year grant funded geometry focused program. The program consisted of two graduate courses, an 11-day geometry focused graduate class from 8:30-12:30 each day during the summer of 2012 and a 7-day geometry focused class from 8:30-4:00 during the 2012-2013 academic year. The researcher centered this study on the former, where participants learned geometry through the use of children’s literature, hands-on activities, and group work. Topics covered during the summer program included polygons, angles, perimeter, area, circles, symmetry, similar figures, and transformations. For each new geometry concept, the researcher, one of three instructors for the course, would read key book excerpts from a children’s book, summarize main story lines, or narrate the entire book to the class as a way to either introduce or recap a hands-on geometry lesson. As part of the course instruction, the researcher read two books in their entirety during the summer term. Approximately 10-20% of the summer class time was spent integrating literature into the curriculum. Some of the books used throughout the summer included such titles as *The Greedy Triangle* (Burns, 1994), *Sir Cumference and the Dragon of Pi* (Neuschwander, 1999), *Sir Cumference and the Great Knight of Angleland* (Neuschwander, 2001), *Sir Cumference and the First Round Table* (Neuschwander, 1997), and *Cut Down to Size at High Noon: A Math Adventure* (Sundby, 2000).

Participants included 18 inservice teachers in a southern state in the United States. During the 2011-2012 academic year, two of the three teachers of the class recruited participants from two area public schools. Participants were selected based on deficiencies in one or more of the following areas: not certified to teach mathematics, less than 24 hours of college level mathematics, little or no college level geometry exposure, alternative certification coupled with less than 3-5 years of teaching experience, and/or less than 1-2 mathematics pedagogy classes. (See Table 1 for more information about participants.)

Table 1

Participant Information

Pseudonym	# of years of teaching experience	Teaching certification
Danielle	.5	Math 4-8
Addison	1	Math 4-8
Drew	1	Generalist 4-8
Matt	2	Math (Probationary)
Ava	2	Math 4-8
Brianna	3	None
Katelyn	3.5	Math 4-8
Kyla	3.5	Math 4-8/8-12, Generalist 4-8
Melody	5	Generalist PK-6/4-8, Special Education (SPED) K-12
Ella	5	Math/Science 4-8
Martha	5	Generalist 4-8
Rachel	6	Math 4-8, ESL
Monique	11	Generalist 4-8
Harry	14	Math 4-8/8-12, SPED K-12
Sydney	15	SPED K-12
Maria	15	Math 4-8
Heather	26	Elementary Math 1-8
Jasmine	26	Generalist 1-8

Data collection consisted of tests, background surveys, journal article reviews, and daily reflections, where daily reflections were the main basis for the findings. Gathering the test and background survey information were for triangulation purposes (Merriam, 1998). On the first day of class, inservice teachers completed a 25-question open-ended geometry test. Participants also completed a technology and a standards-based knowledge survey about their background familiarity with certain types of mathematics technology (e.g., graphing calculators) and standards-based pedagogy (e.g., the use of manipulatives and cooperative learning). Based on their test scores and survey information, the instructors gauged what topics needed more attention than others. Another form of data collection consisted of journal article reviews, where participants reviewed an article of their choosing from NCTM's *Mathematics Teaching in the Middle School*. Lastly, at the end of each class meeting, inservice teachers would utilize a computer lab and reflect through Edmodo, an internet data collection source, where participants were able to respond to 2-3 questions posed by the instructors. While most questions focused on the participants' opinions about daily activities in their usefulness in their classrooms, some were motivated from research, such as the following:

> Explain how you feel the use of children's literature will impact students' understanding of mathematics. Make sure to include any personal experience you may have with utilizing literature in your mathematics classroom or from your schooling (Wilburne & Napoli, 2008).

After collecting all the data, the researcher utilized *Nvivo*, a qualitative software tool, to code the data and search for themes. Open coding techniques (Corbin & Strauss, 2008) allowed the researcher to compile a list of codes for reporting findings.

Findings

Based on data analysis of daily reflections and journal article reviews, the researcher found all participants open to the idea of utilizing children's literature in the classroom. Fourteen of the 18 inservice teachers had already used literature to some extent to teach. For example, Kyla, a teacher for 3.5 years, detailed ways in which fictional books come alive in her classroom.

> My 7th and 8th graders love reading those types of books [mathematics focused] because they get to sit on the floor and talk about what's happening in the book. We also use *Anno's Mysterious Multiplying Jar* (Anno & Anno, 1999) when we start talking about

> factorials, permutations and combinations. My colleagues and I enjoy using as much literature in our classroom as we can.

Even veteran teachers expressed the importance of children's literature in the classroom. Heather, a 6th grade teacher for 26 years, was an advocate of literature and described the impact the grant class has made to her beliefs about literature.

> I have seen several different books here [during the class]. I have also read the *Math Curse* (Scieszka, 1995) to my students, but I have not searched out other books that I can use. Now, I know of several.

Besides discussing ways in which they currently use literature, four teachers remarked on how they might utilize literature in the future as impacted by the class. Brianna, a teacher for 3 years, specifically discussed how the researcher used a book in a unique way that inspired her.

> I feel children's Math Literature will impact their [my students'] understanding by showing examples and pictures that ties in with the lesson that is given to them. For example the story you read was giving us context clues in knowing what we were finding and looking for in the story. Also, this gives the student a chance to figure out the ending of a story line and make a sound judgment on how to approach the problem. I will use more literature in my classes next year to get points across in a more than one way of showing an example.

Despite the positive reactions to literature activities in the course, teachers did express concerns about implementing literature in their teaching. These included lack of experience using literature in the classroom, questions on implementation, and lack of time. For example, Heather candidly discussed her frustrations in a daily reflection:

> I really want to use them [children's books] more this year. Sometimes, I feel like all our time is used up for us by the district. We are given everything to do and we "supplement" on our own but then when they do the tests, it is all over the activities that they gave us. It kind of takes the joy out of it sometimes. I think it stifles teachers. I just shut my door and do my thing! :-)..... I like to make math come alive for the kids.

In addition to exposing four teachers for the first time to fictional literature in the mathematics classroom, five other teachers planned to use children's books more in their own mathematics classes in the future. Rachel, a teacher of 6 years, discussed how she might include reading during the school day.

> I've not used children's literature to teach math, but I have used books before with my 6th graders, and I agree with others in the class that this is a fantastic way to get students excited about a topic. My students would sit (or lay) on the floor while we read and looked at the pictures -- wonderful! I am thinking of doing the type of activity we did today using the Sir Cumference (Neuschwander, 1999) book during our "Tiger Time," which is a half-hour period used for various subjects.

Another indication of the impact of children's literature with the participants comes from the teachers' choice of professional articles to read during the course. For an assignment, participants were to select a geometry related article (from June 2010-present) to read and critique from NCTM's *Mathematics Teaching in the Middle School*. Three teachers of the 18 participants chose *Geometry Sleuthing in Literature* (Wallace, Evans, & Stein, 2011), where the authors describe ways in which to incorporate multiple works of fiction into lessons. Brianna's comments in her review summed up how the use of literature in the grant class has changed her opinion about literature in the mathematics classroom.

> My thoughts after reading this article made me realize the importance of literature in Math. It also showed me another way to reach my students when introducing Geometry concepts to them. This topic was also displayed on Friday in my current math class by the teacher reading to the class and having the students to work out the problem of the story. This was very exciting to me…I will implement more reading of literature involving math concepts and terms to build a better vocabulary and perspective of math…These literatures will be used and discussed in my future classes when Geometry is being taught.

Discussion

The findings from daily reflections and journal article reviews suggest utilizing children's literature in a professional development geometry class positively impacted inservice teachers' beliefs about the importance of fiction in their mathematics classrooms. This open-minded attitude towards reading across the curriculum could be due to the fact that almost all teachers declared in background surveys their standards-based teaching philosophies that included the use of cooperative learning groups and manipulatives in the classroom. Also, a majority of the teachers already had used children's literature as a means of teaching, which could have played a role in their opinions about literature use.

The researcher also found numerous advantages and disadvantages to using children's fiction to supplement the geometry content. One benefit included getting teachers excited about reading in their mathematics classrooms. One teacher brought in an additional text she loved using in her mathematics classroom, which she shared with others. During the academic portion of the grant, another teacher commented on how she decided to use books in her mathematics classes this semester and how surprised and excited the students were to be given the opportunity to learn mathematics in this way.

Even though there are many advantages to utilizing literature in the classroom, one cannot neglect to mention the drawbacks. Some difficulties consisted of having to eliminate other activities to include story time in the agenda, as well as trying to find engaging hands-on mathematics lessons to tie the books into the lessons. The researcher only chose books which could be connected directly to activities conducted in class. This type of daily routine took much time and reflection on the researcher's part, which she felt was well worth the effort.

Based on the findings, college instructors can see the influence of including children's literature in their mathematics instruction to middle school inservice teachers, which in turn can impact the way children learn in middle schools. There are many helpful tricks the researcher found for utilizing children's literature for inservice mathematics teachers. For example, model how to read and how to connect the literature to the concepts. There are numerous online lessons written for specific books available for teacher use, which teachers may be unfamiliar of their existence. Besides online lessons, authors, such as Thiessen (2004), have published books that contain classroom-ready lessons with teacher notes that span many grades and explore numerous book titles. If you do not include such a critical step in the learning process, teachers will often not know how to implement reading into their curriculum. Also, select books that integrate well with course goals and curriculum. This can be a time consuming, but beneficial, part of the preparation time for the class. Another tip is give a list of books to teachers so they can reference any text used in class. Finally, encourage active discussion about the pedagogical features of the literature. This could consist of pausing during certain portions of the book to elicit group discussion or to lead into hands-on activities.

Since this study consisted of only one class of middle school teachers, the findings are limited, but other studies could be conducted in comparable courses. In addition, researchers could extend this inquiry to high school teachers, where the use of children's literature is even less widely accepted and practiced.

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TOWARD IMPROVING MYMATHLAB

Cong-Cong Xing
Nicholls State University
cong-cong.xing@nicholls.edu

DesLey Plaisance
Nicholls State University
desley.plaisance@nicholls.edu

While MyMathLab (MML) has achieved notable success in general mathematics education, it has some fundamental and pedagogical issues that need to be addressed. In this paper, we (1) identify some MML-associated problems that commonly exist in lower-level college mathematics teaching and learning; (2) propose our solution to these problems; (3) demonstrate the solution by examples; and (4) summarize the contribution of this paper. We hope to see an improved MML in the near future.

MyMathLab (MML) is one of the cutting-edge computer-enhanced teaching and learning tools administrated by Pearson Education (2013) where students can practice their mathematics problem solving skills with respect to selected textbooks. It has been widely integrated into lower-level mathematics courses in colleges and universities with some success, as indicated for example in Dries (2012), Livesay (2011), Spence (2007), Rouse (2011), Instructional Media & Magic (2013), and Anderson (2012). However, like many other technological education tools, MML is still in a young state in terms of maturity and needs to be improved, strengthened, and further developed.

In this short paper, based on our professional experiences, we point out two problems that students commonly suffer in dealing with lower-level college mathematics materials. One of the problems is particularly due to MML and is causing MML's mishandling on checking the logical correctness and validity of student work; the other problem refers to the basic understanding or structure of algebraic expressions which is fundamental to all kinds of mathematical work. We propose MML-based pedagogical solutions to both problems.

The rest of the paper is organized as follows: section 2 describes the problematic issues; section 3 provides a proposal for resolving the issues; section 4 gives some illustrating examples; section 5 offers the related literature and theoretical framework; and section 6 concludes the paper.

The Issues

We address two issues that we have noticed throughout the years of teaching. All examples shown in this section are samples of authentic student work collected by the authors in teaching a course titled Calculus with Business and Economic Applications. (This course was used because it is taught by both authors and the use of MML was mandated by their University department.)

"Answers Only" Needs to be Changed

For most lower-level mathematics textbooks/courses (e.g., Sullivan (2010), Lial, Hungerford, and Holcomb (2011), and Pirnot (2009)), Pearson offers a matching MML component which contains, among other things, the exercises at the end of textbook chapters/sections allowing students to complete their homework and tests on line. Unfortunately, for the majority of these exercises, MML only requires the submission of answers. That is, as long as the final answer submitted by a student to a problem is correct, then the student will get full credit for that problem regardless of the solution process, and even if the solution process is seriously flawed. We give two examples here to show this point. (Throughout the paper, limit properties refer to Constant Property, Identity Function Property, Sum Property, Difference Property, Product Property, Quotient Property, Power Property, and Polynomial Property which can be found in any standard calculus-related textbooks.)

Example 1. Use limit properties to find

$$\lim_{x\to 4}\left(\sqrt{x}+1\right).$$

Student solution:

$$\begin{array}{lll} & \lim_{x\to 4}\left(\sqrt{x}+1\right) & \\ (step\ 1) & =\sqrt{4}+1 & (Polynomial\ Property) \\ (step\ 2) & =2+1 & (Algebra) \\ (step\ 3) & =3 & (Algebra) \end{array}$$

Here, although the answer (integer 3) is correct, the solution process is *fundamentally* wrong: $\sqrt{x}+1$ is not a polynomial and thus the polynomial property cannot be applied to $\lim_{x\to 4}\left(\sqrt{x}+1\right)$ in the first step. Students who make this kind of mistake clearly do not have conceptual understanding of polynomials; MML cannot identify this conceptual misunderstanding in that all MML expects from the student is the integer 3. While there may be more than one way to solve this problem, the following is an example of a valid solution.

Valid Solution:

$$\lim_{x \to 4} \left(\sqrt{x} + 1 \right)$$

$$(step\ 1) \qquad = \lim_{x \to 4} \left(x^{1/2} + 1 \right) \qquad (Algebra)$$

$$(step\ 2) \qquad = \lim_{x \to 4} x^{1/2} + \lim_{x \to 4} 1 \qquad (Sum\ Property)$$

$$(step\ 3) \qquad = (\lim_{x \to 4} x)^{1/2} + 1 \qquad (Power\ \&\ Consant\ Properties)$$

$$(step\ 4) \qquad = 4^{1/2} + 1 \qquad (Identity\ Function\ Property)$$

$$(step\ 5) \qquad = 3 \qquad (Algebra)$$

Example 2. Use limit properties to find

$$\lim_{x \to 1} (x^2 + 1).$$

Student solution:

$$\lim_{x \to 1} (x^2 + 1)$$

$$(step\ 1) \qquad = \lim_{x \to 1} x^2 + \lim_{x \to 1} 1 \qquad (Sum\ Property)$$

$$(step\ 2) \qquad = \lim_{x \to 1} (x)^2 + 1 \qquad (Power\ \&\ Constant\ Properties)$$

$$(step\ 3) \qquad = 1^2 + 1 \qquad (Identify\ Function\ Property)$$

$$(step\ 4) \qquad = 2 \qquad (Algebra)$$

Here, again, although the answer (integer 2) is correct, the solution process is fundamentally faulty. In the second step, when the Power Property is applied to $\lim_{x \to 1} x^2$, the result should be $(\lim_{x \to 1} x)^2$, not $\lim_{x \to 1} (x)^2$. (Note that this is not a careless writing mistake made by the student as both authors have needed to go through multiple discussions and clarifications with their classes regarding this issue.) Students who write this kind of solutions have deep trouble in differentiating the semantics of the following expressions

$$\lim_{x \to 1} x^2 \ , \qquad \left(\lim_{x \to 1} x \right)^2 \ , \qquad \lim_{x \to 1} (x)^2$$

and do not comprehend the essential meaning of the Power Property. Moreover, step 3 is also incorrect because $(x)^2$ which means the same as x^2 is not in the form of the identity function (body) and Identity Function Property thus cannot be used in this step. Obviously, errors shown here are of basic but foundational algebraic concepts, and the answer obtained in this way makes no mathematical sense and should not be given credit by MML. The correct solution is given below.

Valid solution:

$$\lim_{x\to 1}(x^2+1)$$

$$
\begin{array}{lll}
(step\ 1) & = \lim_{x\to 1} x^2 + \lim_{x\to 1} 1 & (Sum\ Property) \\
(step\ 2) & = (\lim_{x\to 1} x)^2 + 1 & (Power\ \&\ Constant\ Properties) \\
(step\ 3) & = 1^2 + 1 & (Identify\ Function\ Property) \\
(step\ 4) & = 2 & (Algebra)
\end{array}
$$

Structures Embedded in Expressions

Many students have little awareness for the structure of mathematical expressions which is regularly indicated by the precedence of operators and pairs of parentheses "(" and ")" or square brackets "[" and "]", and are unable to recognize the significant difference between the presence and absence of these delimiters. A typical example is as follows.

Example 3. Use limit properties to find

$$\lim_{x\to 0}[x(x+1)].$$

Student solution:

$$\lim_{x\to 0}[x(x+1)]$$

$$
\begin{array}{lll}
(step\ 1) & = \lim_{x\to 0} x \cdot \lim_{x\to 0}(x+1) & (Product\ Property) \\
(step\ 2) & = \lim_{x\to 0} x \cdot \lim_{x\to 0} x + \lim_{x\to 0} 1 & (Sum\ Property) \\
(step\ 3) & = 0 \cdot 0 + 1 & (Identity\ Function\ \&\ Constant\ Properties) \\
(step\ 4) & = 1 & (Algebra)
\end{array}
$$

The crucial error in this work is that there must be a pair of ( ) around $lim_{x\to 0}\, x + lim_{x\to 0}\, 1$ in the second step which is unfortunately missing. And the absence of this pair of ( ) inevitably leads to an incorrect result. This work is a disaster in the sense that both solution process and the answer are wrong. Students who make this kind of mistake fail to recognize that the structure of the expression $lim_{x\to 0}\, x \cdot lim_{x\to 0}(x+1)$ in step 1 is a product of two sub-expressions, and once the second sub-expression is expanded into the sum of two smaller sub-expressions, this expression is *still* a product of two (slightly more complex) sub-expressions. That is, the top-level structure of the expression in step 1 must be *preserved* in step 2 (and in subsequent steps). The correct computation of this problem should be as follows.

Valid solution:

$$\lim_{x \to 0}[x(x+1)]$$

$$\begin{array}{llr} (step\ 1) & = \lim_{x\to 0} x \cdot \lim_{x \to 0}(x+1) & (Product\ Property) \\ (step\ 2) & = \lim_{x\to 0} x \cdot (\lim_{x\to 0} x + \lim_{x\to 0} 1) & (Sum\ Property) \\ (step\ 3) & = 0 \cdot (0+1) & (Identity\ Function\ \&\ Constant\ Properties) \\ (step\ 4) & = 0 & (Algebra) \end{array}$$

Part of the reasons that cause this type of mistake is the way in which mathematical expressions are written traditionally — all symbols, numbers, parentheses, operators, notations, etc. are put in one single line in a linear order. As such, the structure of a mathematical expression which is critical for the (understanding of the) meaning of the expression is actually embedded in this linear form, and thus may not be easily seen.

Our Proposal

We now propose our solutions to the issues raised in this paper.

"111" on MML

For the problems described in section "'Answers Only' Needs to be Changed", we suggest that MML requires, for most exercise problems, the submission of not only the answer but the solution process as well. Although letting computers grade solution processes for mathematical problems is a challenging task, once the scope of the problem is restricted to certain types, the task should be manageable. Specifically, for problems of finding limits using limit properties, we suggest that MML

- Requires students write out both computation steps (including the final answer of course) and the names of the appropriate properties used in the computation.
- Uses the common "two-column" format in this matter. That is, computation steps will be written in the left column and names of properties will be written in the right column. This "two-column" setting is a standard format in the teaching and learning of calculus and general mathematics, and has been practiced using "pencil-and-paper" all the time. Now, it is time to see this activity on MML.

- Adopts the "111" (a term coined by the authors of the paper and explained below) style when computerizing the grading of solution processes. "111" refers to a particular style of limit computation using limit properties which intends to foster students to develop an ability to solve problems in a gradual, firm, and orderly manner, and to make the computerization of grading solution processes easier to implement. It denotes the following: for each *one* (1) computation step, choose only *one* (1) property, and apply that property only *once* (1).

Limit Computation Trees

For the problem described in the section "Structures Embedded in Expressions", we propose an alternate method to the traditional line-by-line linear form for conducting and presenting the computation of limits — *limit computation trees*. The scheme of such computation trees is shown in Figure 1 where *exp* is a mathematical expression; *subexp1* and *subexp2* are its sub-expressions; *prop1* to *prop3* are instances of limit properties; *op1* to *op3* are mathematical operators ($+, -, \times,\ /$ etc.); and $n1$ to $n4$ are numbers.

The essential idea of the limit computation tree is to show exactly how the process of computing a limit, which is entirely driven by repeatedly applying various limit properties, starts, proceeds, and terminates by displaying the structure of and the correlation between all sub-computations (induced by the original computation) in a top-down graphical manner. An example of the limit computation tree is given in the next section.

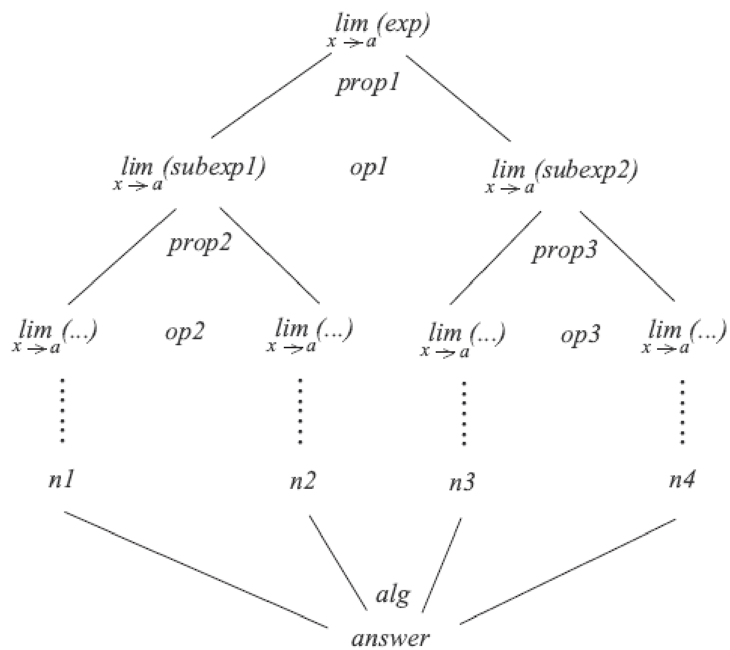


Figure 1: Limit Computation Tree

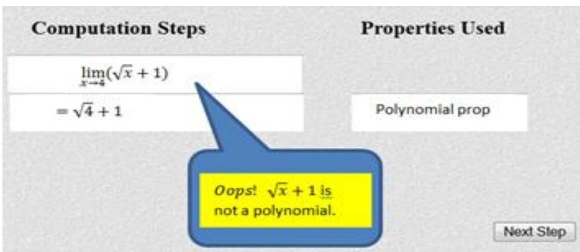
Illustrations

We now demonstrate how problems identified previously in the paper can be resolved by using the methods proposed in the last section. Regarding the "answers only" problem and the proposed solution for it, MML may set up a framework depicted in Figure 2(a) where students type the result of applying a certain limit property in the "Computation Steps" field and the name of that property in the "Properties Used" field, and click the "Next Step" button to try to proceed. If the contents in both fields are correct, then clicking the "Next Step" button will generate a new row with these two fields below the current row to allow students to continue the work; otherwise, this clicking action will generate an error message alerting what has gone wrong. The incorrect computation and the correct computation discussed earlier in the second section should be caught and approved by MML as shown in Figures 2(b) and 2(c), respectively.

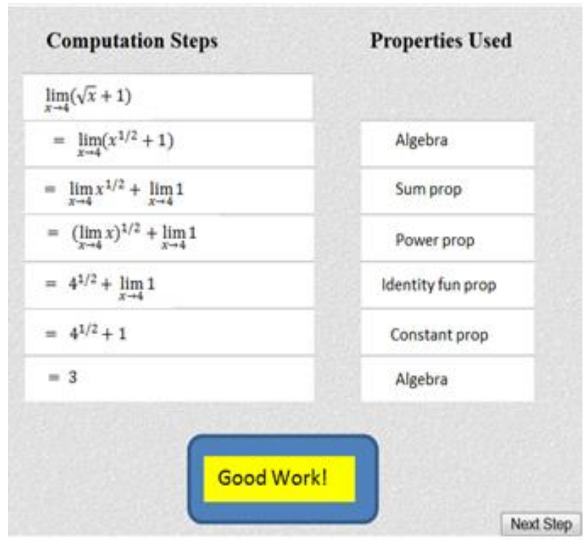
For the algebraic expression structure problem and the proposed solution for it (Limit Computation Tree), MML may devise a page similar to Figure 3(a) where the limit computation problem can be entered. Figures 3(b) through 3(h) demonstrate the stages of the construction of the limit computation tree for $\lim_{x\to 0}[x(x+1)]$ when it is entered in Figure 3(a).

Specifically, clicking the "Click to See the Computation Tree Step by Step" button in Figure 3(a) leads to Figure 3(b); clicking the node containing $\lim_{x\to 0}[x(x+1)]$in Figure 3(b) leads to Figure 3(c) which denotes that the Product Property has been applied to $\lim_{x\to 0}[x(x+1)]$ and

(a) MML Setting for Limit Computation



(b) Wrong Computation Caught.



(c) Correct Computation Confirmed.

Figure 2: "111" Limit Computation on MML.

the result is $\lim_{x\to 0} x+\lim_{x\to 0}(x+1)$; clicking the node containing $\lim_{x\to 0} x$ in Figure 3(c) leads to Figure 3(d) which shows that $\lim_{x\to 0} x$ has been resolved to 0. Once all sub-limit computations are resolved (to numbers), a button showing "Click to See Final Answer" will appear on the page (see Figure 3(g)) and clicking this button will produce the final answer (see Figure 3(h)). As we can see in Figure 3(e), the lower $\lim_{x\to 0} x$ is not connected to the upper $\lim_{x\to 0} x$ and cannot operate with the upper $\lim_{x\to 0} x$, thus preventing the student error shown in the previous section from occurring here.

Related Literature, Theoretic Framework, and Discussion

As we have seen, MML's "answers only" situation needs to be changed. This is also evident from the research in mathematics education and pedagogy. For example, Idris (2009, p.36) stated that "learning currently no longer emphasizes correctness of the final answer but has shifted to emphasizing process, context, and understanding." Besides the algebraic work embedded in these limit computations, the notion and notation of limit itself may also contribute to the students' mistakes as limit is a typically formidable task for students to grasp (Davis and

Vinner (1986)). Thus, it is important that MML addresses the “right” answer with a wrong process issue in order to not foster mishandlings within the study of limits.

Note that tree diagrams have been used by several mathematics education researchers (e.g., Ernest (1987), Kirshner and Awtry (2004), Sleeman (1984), and Thompson and Thompson (1987)) to illustrate the idea of operation precedence within algebraic expressions (the lower an operation appears in a tree, the higher its precedence is), and that the computation of the

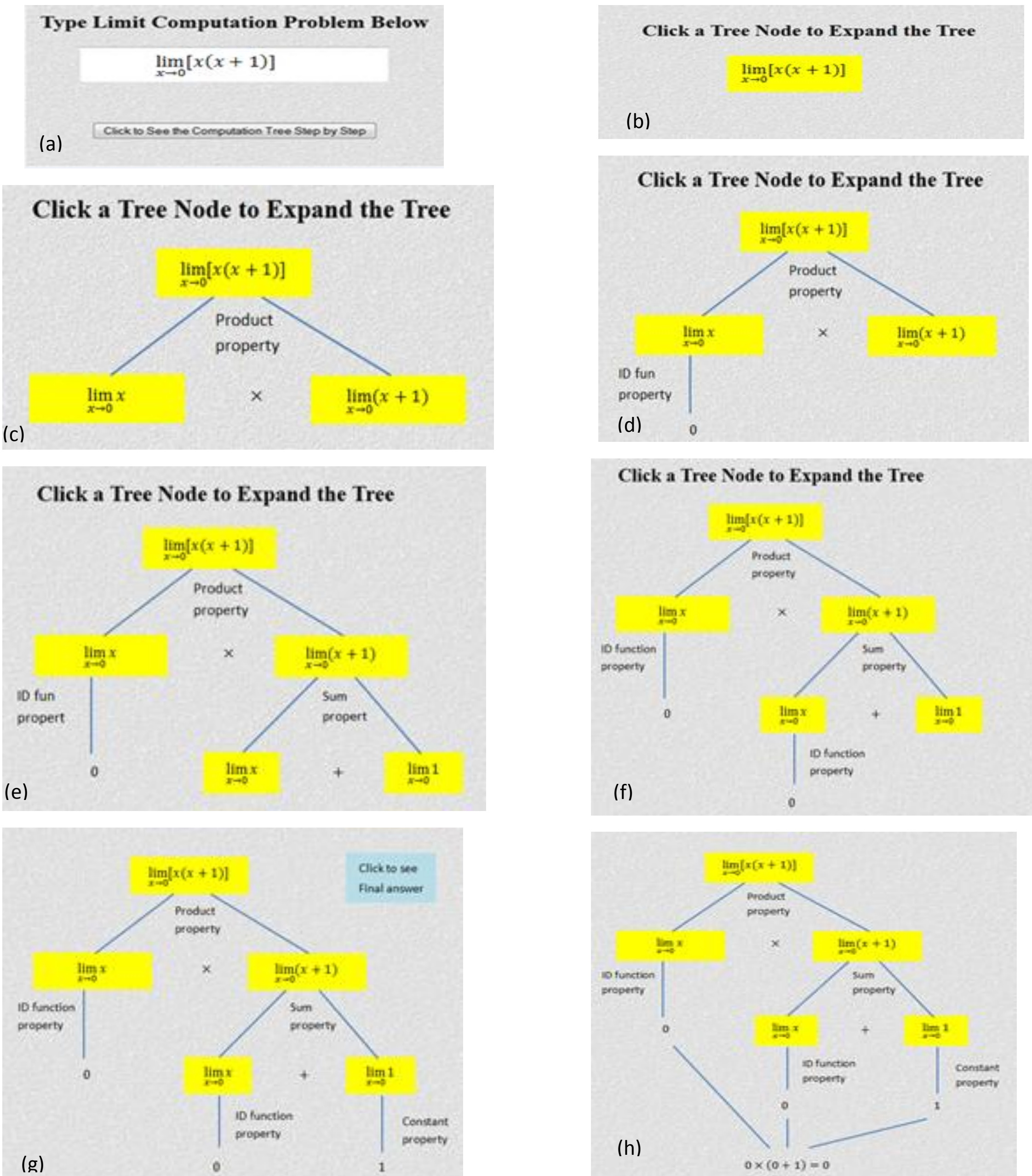


Figure 3: An Example of Limit Computation Tree.

algebraic expression is carried out from the bottom of the tree toward the top of the tree (bottom-up). In this sense, the limit computation tree has an "opposite" meaning to that of the algebraic expression trees as the first limit property to be chosen to apply to the expression is determined by the operation with the least precedence in that expression. That is, the computation of the limit of an expression starts from the top of the tree and moves downwards (top-down).

We believe that the students' trouble in understanding the structure of math expressions is related to their reading habit. There is an obvious disparity in the reading of symbols/math expressions and prose. Students are accustomed to reading prose from left to right and top to bottom. Quinnell and Carter (2012, p.36) stated, "…. students must use judgment or experience in deciding how to read a symbol. This could be why some students manage to grasp the subtleties of decoding symbols, while others remain perplexed."

Regarding the popularity and claimed success of MML (see resources mentioned at the beginning of the paper), we do note that in most of those cases, students' success is defined by their completing a course with a grade of A, B, or C, and most of the results were reported via venue associated with Pearson Education. Southeast Comprehensive Center (SECC) of SEDL (2011) conducted a search to collect studies related to the effectiveness of mathematics software products; no MML research studies were located. Nevertheless, a study by Kodippili and Senaratne (2008) does indicate a strong MML-related student success rate, although the authors admit some methodological shortcomings.

Concluding Remarks

Mathematics learning, especially problem solving, is not just about finding the (correct) answers. Rather, gaining a substantial understanding for relevant concepts and subsequently applying this understanding logically in problem solving is more important than the answer itself. MML, as the leading computer software for mathematics education on the market, unfortunately takes an unsound approach in that all it requires from the student for problem solving is a correct answer. In order to address this issue, we have proposed two specific solutions in this paper toward improving MML. We believe that the paper makes the following contributions:

- The "111" style for limit computation, which can be easily implemented on MML, rectifies MML's "answers only" problem.

- The limit computation tree, which can also be implemented on MML, can help students truly comprehend the process of computing limits by using limit properties.

To further evaluate MML, and as future research work, we may focus on the performance of students in subsequent courses (mathematics, business, engineering, science, etc.) in which MML mathematics courses are necessary prerequisites. In that the success of students in MML courses is decided by their earning a grade of A, B, or C, there is no evidence that students will be successful in subsequent courses relying on the application of the previously learned mathematics.

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EXPLORING TEACHERS' CATEGORIZATIONS FOR AND CONCEPTIONS OF COMBINATORIAL PROBLEMS

Nicholas H. Wasserman
Southern Methodist University
nwasserman@smu.edu

While counting is simple enough, counting problems span the spectrum of difficulty. Although mathematicians have succinct categories for differing problem types, students struggle to model solving problems and to identify related problem structures. In a graduate course, K-12 mathematics teachers (n=7) were introduced to combinatorial problems and then given a set of problems to solve and categorize. Results from this study specify ways that mathematics teachers who are also novice combinatorialists identified similarities between problems; two particularly difficult problems reveal poignant conceptions and explanatory categorizations.

With increased emphasis on probability in K-12 mathematics education (e.g., Common Core State Standards, 2010), knowledge of combinatorial thinking is becoming increasingly necessary for both students and teachers (e.g., counting the cardinality of sets and sample spaces). Permutations and combinations, while frequently included in the curriculum, are often tangential topics in the scope of mathematics learning and only superficially discussed. Kapur (1970) noted potential benefits for integrating combinatorics into the K-12 curriculum, which include making conjectures, thinking systematically, one-to-one mappings, and many applications in physics, biology, and computer science. The rapid pace and content coverage required for state exams may be one source of blame for the current disintegration; however, another probable reason is a lack of knowledge or comfort with combinatorics on the part of teachers. Counting problems can be very challenging, and, while expert mathematicians have succinct categories for differing problem types, the process for learning to think combinatorially may not be so neatly packaged. This paper looks at categorizations for and conceptions of different combinatorics problems made by middle and secondary mathematics teachers; interesting findings and implications for the learning and teaching of combinatorial problems are discussed.

Literature

While counting is simple enough, counting problems span the spectrum of difficulty. Even authors of combinatorics textbooks weigh in on the difficulties encountered and insights required in such problems (e.g., Tucker, 2002). One issue in learning combinatorics is finding appropriate ways to model specific problems. Batanero, Navarro-Pelayo, & Godino (1997) discuss three different implicit models – selections, distributions, and partitions – for combinatorial problems;

furthermore, each model may result in different solutions based on other structures within in the problem. Identifying common structures, beyond modeling, within otherwise dissimilar problems also serves as a barrier to the learning and teaching of combinatorics (e.g., English, 1991) – despite the fact that expert mathematicians have identified nice categories for different counting problems according to the common 2x2 matrix of: with and without repetition, and ordered and unordered selection (see Table 1). While many problems may require more than one of these four approaches, even the basic distinctions between these problem types may not be fully understood by novice learners, particularly given the variety of modeling techniques. In fact, the connections (or lack thereof) made by novices as they solve counting problems can provide insight into common conceptions and misconceptions faced during the learning process.

Table 1
Selecting k objects from n distinct objects

	Ordered (permutations)	**Unordered** (combinations)
Without repetition	Arrangements $\frac{n!}{(n-k)!} = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1)$	Subsets $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
With repetition	Sequences $n^k = \underbrace{n \cdot n \cdot n \cdot \ldots \cdot n}_{k}$	Multisubsets $\left(\binom{n}{k}\right) = \binom{k+n-1}{n-1}$

Adapted from: Benjamin, A.T. (2009, p. 10)

Identifying ways to apply knowledge from previously learned problems to another context is generally known as *transfer*. The roots of transfer extend back to behaviorism, where the idea was viewed as fundamental to the learning process. More recently, however, alternatives and adaptions to the traditional view of transfer have been articulated; in particular, Lobato (2003) characterizes *actor-oriented transfer* (AOT). AOT shifts the perspective regarding transfer from an expert's view to a learner's vantage point, which results in paying particular attention to the ways that novices draw on their knowledge to solve new problems. Lockwood (2011) argues that AOT is a particularly poignant perspective for investigating combinatorial learning because the subject depends strongly on "establishing structural relationships between problems" (p. 309). Given the importance of combinatorial thinking for and the current emphasis on understanding probability and statistics, efforts using AOT to investigate how such thinking develops, for students and teachers, are warranted. Specifically, this paper addresses the following question: How do middle and secondary mathematics teachers who are also novice combinatorialists categorize and conceptualize different combinatorial problems?

Methodology

As a starting point for investigating how middle and secondary mathematics teachers that are novice combinatorialists categorize and conceptualize various problem types, two focus groups (e.g., Berg & Lune, 2012) were conducted (N=3 and N=4). The focus groups were conducted in conjunction with a graduate mathematics education course; all seven participants in the focus groups were practicing middle and secondary teachers with less than 6 years teaching experience and were enrolled in the course. The focus groups were preceded by a brief introduction to combinatorics problems in the course. While the middle and secondary teachers in the course had various mathematical backgrounds, none of the focus group participants had completed a course in combinatorics or discrete mathematics, making them novice combinatorialists.

The brief introduction (~90 minutes) in the course consisted of two parts: 1) overt instruction on the addition principle; the multiplication principle; factorial notation; and dividing out extraneous solutions when order is irrelevant, including the $\binom{n}{k}$ notation; and 2) approaches and solutions to six combinatorics problems, which were selected as being relatively common examples of the four types of problems from textbooks and other literature. The six problems (see Table 2) were presented to students as a way to expose them to various strategies for solving combinatorial problems; no structural characteristics of problems (e.g., order matters, repetition allowed) were mentioned and no connections between "types" of problems were discussed.

Table 2
Description of Combinatorics Problems presented to participants with solutions

Name	**Description**	**Type & Solution**
Handshake	If 10 people are at a party and everyone shakes hands with everyone else, how many total handshakes are given?	Subset $\binom{10}{2}$
Password	A password has to be 8 characters long and can use any of the 26 letters or the 10 digits (not case sensitive). How many different passwords are there?	Sequence 36^8
Hot Dogs	Hot dogs come in 3 varieties: Regular, Chili, Super. How many different ways are there to purchase 6 hot dogs?	Multisubset $\left(\binom{3}{6}\right) = \binom{6+2}{2}$
Voting	Two candidates are running for a club election. In the end, candidate A gets 4 votes and candidate B gets 5 votes. The moderator of the club, however, reads each vote out loud in order. How many different ways could he read out the votes?	Subset $\binom{9}{4} = \binom{9}{5}$
States	How many different "words" can you make with the letters (nonsense words count) in TEXAS? How about in MISSISSIPPI?	Arrangement $5!$ $\frac{11!}{4!4!2!}$
Vowel	You are creating 5 letter words that CAN repeat letters. How many words are there that have at least one vowel?	Sequence $26^5 - 21^5$

After instruction in the course, the study participants (N=7) were randomly assigned to one of two focus groups (~120 minutes each), during which they worked together on an assortment of 12 combinatorial problems, ranging in type and complexity. Through the lens of AOT, participants were asked to: "Answer each of the problems and organize them into 'groups' of problems that have similar methods for solving. For each group of problems, provide a brief description of how and why the problems in that group are similar." The twelve problems, along with the six original problems discussed in class, were printed on note cards to facilitate participants' groupings. While focus group participants worked on the problems and discussed ideas with one another, the researcher took field notes about important comments or connections made by participants (i.e., occurrences of AOT), at times asking questions to uncover their thinking. Participants' mathematical work and their final groupings/descriptions were collected for the study. For space purposes, only some of the problems are described in detail as they come up in the discussion and analysis; however, all problems are listed in Table 5 in the Appendix for reference.

Findings

The categorizations and descriptions created by two focus groups of middle and secondary mathematics teachers provide some information regarding AOT in the learning of counting problems. Generally, participants were able to make and describe the structural connections about *permutations* (Arrangements and Sequences), where order matters, much easier than *combinations* (Subsets and Multisubsets), where order does not matter. With the exception of the Vowel problem, which involved subtracting two sequences, groups were able to identify 100% of the possible Arrangement and Sequence problems (Table 3). The groups, however, also placed extra problems in these categories (reasons are discussed later). In addition, the focus groups were able to portray the structural similarities between these two problem types precisely: both descriptions explicitly state the appropriate characteristics related to order and repetition.

Table 3
The groups' categories and descriptions for permutation problems

Type	Problems	Group 1		Group 2	
Arrangements (Ordered, without repetition)	Problems States Netflix Plane Routes	Problems States (Texas) Netflix Plane Routes	Description -Each thing can only be in one place at a time (no repeats within set) -Order matters	Problems States Netflix Plane Routes *M/F Committees*	Description -The order of choices matters. -Choices cannot be repeated.
Sequences (Ordered, with repetition)	Problems Password MC Exams1 Gift Cards 4-letter words Vowel	Problems Password MC Exams1 Gift Cards 4-letter words [in Cases] *Marbles*	Description -Things being distributed to different positions -Order matters -One element can be repeated (people getting more than one card)	Problems Password MC Exams1 Gift Cards 4-letter words [missing]	Description -Certain amount of spaces and each space has the same number of options. -Options can be repeated.

The two focus groups had much more difficulty categorizing and describing *combination* problems (Subsets and Multisubsets). While an apparent difference exists between the two groups in their ability to identify common structures between Subset problems (Group 2 successfully accounted for 4 of the 5), both groups had particular difficulty with Multisubset problems (Table 4). Group 1 was unable to solve any of these problem types: indeed, the Marbles problem was incorrectly solved as a Sequence and the Hot Dogs problem, which was solved in class, was connected to simple Subset problems (the group did not appreciate the unique characteristics of the original problem, which was then translated through a stars and bars model to a simple Subset problem). Group 2 solved the Skittles problem and was able to connect it to the Hot Dogs problem; however, rather than focusing on the similar characteristics of these problems, their description for this category was procedural (see Table 4), which demonstrates less sophisticated expertise (e.g., Schoenfeld & Hermann, 1982) and indicates that learners may have difficulty identifying common structures within combination problems.

Table 4

The group's categories and descriptions for combination problems

Type	Problems	Group 1		Group 2	
Subsets (Unordered, without repetition)	Problems Handshakes Supreme Court Voting MC Exams2 M/F Committees	Problems [missing] Supreme Court Voting [did not do] [did not do] *Hot Dogs*	Description -How many different positions each element can occupy (9 choose 6)	Problems Handshakes Supreme Court Voting MC Exams2 [in Arrangement]	Description -Take the groups and choose a certain number. Using the "group" choose "number" gets rid of the duplicates. The duplicates exist because order does not matter.
Multisubsets (Unordered, with repetition)	Problems Hot Dogs Summed Digits Skittles Marbles Pizza Toppings	Problems [in Subset] [did not do] [did not do] [in Sequence] [did not do]		Problems Hot Dogs [did not do] Skittles [did not do] [missing]	Description -Broke into groups to account for no duplicates. The barriers separated into groups. Barriers made choosing easy and allowed for choosing all of one type.

Lastly, Group 1 had an additional category; the two problems identified in this category, Vowel and MC Exams2, were characterized as a "Way to choose a minimum number of outcomes (1 vowel, 2 vowels, etc.). Elements have different characteristics, different groupings. Must look at characteristics as a subset of population." In other words, they viewed problems as similar if they were best solved by splitting into "cases," which, for both problems, was an accurate statement and approach. This gives an indication that, at times, participants made connections according to similar processes instead of structurally similar characteristics.

Discussion

While many findings could be explored in more detail, we will focus on the insight gained from two particularly difficult problems: Gift Cards and Pizza Toppings. These results from the focus groups potentially shed some light on the learning and teaching of combinatorics problems.

The Preferred Vantage Point

The Gift Cards problem (i.e., How many ways can you distribute a \$1, \$2, \$5, \$10, and \$20 gift card to 8 friends?) is a Sequence problem, which, overall, students were able to solve. However, as an individual case, this problem caused surprising difficulty. (Solving the problem also caused over-generalization to other problems with repetition: "like the Gift Card problem," Group 1's reason for including the Marbles problem as a Sequence.) Both groups began by drawing eight slots, one for each person. Their attempts to distribute the five gift cards to these eight people included, among others, $\binom{8}{5}$ (but then "a person could get more than one gift card") and 5^8 (but then "the last person would not have five choices"). Trying to count which person receives which gift card(s) causes modeling difficulties: each person could have anywhere from 0 to 5 gift cards, and sequential models (i.e., eight slots) make the result for subsequent persons dependent on previous ones. To solve it from this perspective would require accounting for each of the seven distinct integer partitions of 5, and then distributing the gift cards according to these possible partitions, which becomes quite complex. It was not until the participants shifted from the *perspective of the people*, who are receiving gift cards, to the *perspective of the gift cards*, which are being distributed, that progress was made. This shift requires accounting for five gift cards (not eight people): each gift card can be given to any one of eight people (i.e., 8^5). However, taking the perspective of a gift card, as opposed to a person, is less natural – I could care less about to whom *every gift card* gets distributed than to which gift cards *I* am going to receive. The exceptional difficulty encountered by initially modeling the problem from the people's perspective may provide some implications for the teaching and learning of counting problems. In particular, given that counting problems can frequently be modeled from both of two different perspectives, there seems to be a potential limitation or misconception associated with *the preferred vantage point*, characterized by novices having difficulty modeling combinatorics problems from the less natural (but combinatorially easier) perspective.

Another Approach To Multisubset Problems

The Pizza Toppings problem (i.e., How many ways are there to make a pizza with 2 toppings, if the choices were pepperoni, olives, sausage, ham, mushrooms, and anchovies (double toppings allowed)?), technically, is a Multisubset problem, $\left(\binom{6}{2}\right)=\binom{2+5}{5}$, with repetition and unordered selection. However, participants split it into the sum of two Subset problems: two different toppings, $\binom{6}{2}$, and two identical toppings, $\binom{6}{1}$. In fact, this solution is insightful because it mirrors their (unsuccessful) attempts at solving other Multisubset problems, such as the Summed Digits problem (i.e., How many numbers between 1 and 10,000 have the sum of their digits equal to 9?). Participants tried to simplify by first selecting one, two, three, or four place values (Thousands, Hundreds, Tens, and Ones) on which to distribute the sum of 9 (the leftover place values being assigned a zero). For example, if you only choose one place value, $\binom{4}{1}$, then there is only one way to produce a sum of 9 for each (i.e., 9000, 0900, 0090, 0009); however, if you choose three place values, $\binom{4}{3}$, then the sum of 9 can be accomplished by accounting for the partitions of 9 that use three values (i.e., (7, 1, 1), (6, 2, 1), (5, 2, 2), (5, 3, 1), (4, 3, 2), (4, 4, 1), (3, 3, 3)) and ordering those partitions to account for repeated values. This model for solving the Summed Digits problem was the participants' natural approach, though quite complex. In fact, after my own investigation, all Multisubset problems can be solved using this approach – although the numerous computations quickly become burdensome. The general solution to a Multisubset problem, $\left(\binom{n}{k}\right) = \binom{k+n-1}{n-1}$, can be proved to be equivalent to: $\sum_{i=0}^{k-1}\binom{n}{k-i}\binom{k-1}{i}$. The first term in this sum accounts for the various cases, e.g., $\binom{6}{2},\binom{6}{1}$ in the pizza problem or $\binom{4}{4},\binom{4}{3},\binom{4}{2},\binom{4}{1}$ in the Summed Digits problem, and the second term quantifies the different ways each of those cases can occur within a given problem.

Conclusion

The findings from this study indicate that learners are able to structurally connect and characteristically conceptualize *permutation* problems (ordered selections) with more ease than *combination* problems (unordered selections). Likely, the sequential modeling of problems, which frequently is useful and naturally lends itself to ordered selections, may contribute to the difficulty accounting for unordered selections. The *preferred vantage point* for modeling may also limit novices' abilities to solve counting problems; teachers should be aware of both perspectives and a learner's tendency toward the more natural (or preferred) perspective. Multisubset problems were found to be the most difficult to solve; indeed, unordered selection with repetition requires a fundamental reconception about the problem. For example, for the Summed Digits problem to emphasize unordered selection with repetition would give peculiar solutions like HTTTHOThOH (i.e., 1,332). Participants' work on the Pizza Toppings problem also indicates a different way to approach solving Multisubset problems, potentially more aligned with novices' development. While the counting computations in this method become increasingly prohibitive, the process could be used as a transitional stage that provides students with a natural way to connect to the problem and increasingly moves toward more efficient algorithms. Overall, the perspective from middle and secondary teachers within this study presents some ideas about the learning and teaching of counting problems that, while not claimed with absolute certainty, are of interest and merit further exploration and investigation.

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APPENDIX

Table 5

Description of Combinatorics Problems in Focus Group for participants to solve

Name	Description	Type & Solution
MC Exams1	An exam contains 15 multiple-choice questions, each with 4 choices. How many possible ways of answering these 15 questions are there?	Sequence 4^{15}
Plane Routes	A plane starts in New York City and will travel to 7 different cities before it returns. How many different ways can the plane do this?	Arrangement 7!
Supreme Court Decisions	In how many different ways can the nine members of the Supreme Court reach a six-to-three decision?	Subset $\binom{9}{6}=\binom{9}{3}$
Summed Digits	How many numbers between 1 and 10,000 have the sum of their digits equal to 9?	Multisubset $\left(\binom{4}{9}\right)=\binom{9+3}{3}$
Gift Cards	How many ways can you distribute a $1, $2, $5, $10, and $20 gift card to 8 friends?	Sequence 8^5
Skittles	16 skittles go into the small Halloween skittle bags. There are 5 colors to choose from in each bag – Red, Green, Yellow, Orange, and Purple. How many different possible bags of skittle are there?	Multisubset $\left(\binom{5}{16}\right)=\binom{16+4}{4}$
The M/F Committees Problem	There are 7 women and 4 men in a club. How many different 4-person committees have at least two women?	Subset $\binom{7}{2}\binom{4}{2}+\binom{7}{3}\binom{4}{1}+\binom{7}{4}$
Netflix	You have 24 different movies on your Netflix account. In how many different ways could you order them?	Arrangement 24!
MC Exams2	An exam contains 15 multiple-choice questions, each with 4 choices. In how many of the possible ways to answer the exam are at least 10 correct?	Subset $\sum_{k=10}^{15}\binom{15}{k}$
4-letter words	Suppose you make a 4-letter "word" (nonsense words count) from the letters A, B, C, D, and E, where you can repeat letters. How many different "words" are possible?	Sequence 5^4
Marbles	How many ways are there to distribute 25 indistinguishable marbles into 7 different containers?	Multisubset $\left(\binom{7}{25}\right)=\binom{25+6}{6}$
Pizza Toppings	How many ways are there to make a pizza with 2 toppings, if the choices are pepperoni, olives, sausage, ham, mushrooms, and anchovies (double toppings allowed)?	Multisubset $\left(\binom{6}{2}\right)=\binom{2+5}{5}$

THE PRIMACY OF FRACTION COMPONENTS IN ADULTS' NUMERICAL JUDGMENTS

Thomas J. Faulkenberry
Texas A&M University - Commerce
Thomas.Faulkenberry@tamuc.edu

Sarah A. Montgomery
Texas A&M University - Commerce
Sarah.Montgomery1990@gmail.com

A fundamental question in numerical cognition concerns how people make judgments about the magnitude of fractions. There is much debate around the issue of whether fraction representations are holistic or component-based. In the present study, we measured hand movements as people mentally compared fractions to ½. We found that participants' hands tended to move according to the size of components rather than the overall magnitude of the fraction. This indicates that people form an initial automatic representation that is tied to surface format (i.e., component-based), but later refine this representation according to task demands.

When skilled adults think about fractions, what do their representations look like? For instance, suppose you were asked to compare the fraction 3/7 to ½. Which is bigger? How do you make the decision? While there are multiple representations that can be deliberately formed depending on context (Lamon, 2005), we are interested in the automatic, unconscious mental representations that are formed when comparing fractions. The issue is by no means trivial: many recent studies have yielded equivocal results with respect to this issue. On one hand, some researchers believe that mental representations of fractions are based on the components of the fraction instead of the fractions numerical value, or magnitude (Bonato, Fabbri, Umilta, & Zorzi, 2007). On the other hand, others have found that people tend to immediately process the magnitude of fractions rather than the components (Meert, Gregoire, & Noel, 2009; 2010; Schneider & Siegler, 2010). Recent evidence indicates that the true answer may lie somewhere in the middle: Faulkenberry and Pierce (2011) concluded that the type of representation formed in a fraction task depends heavily on the strategy used.

The fundamental question becomes the following: in the 1-2 seconds that it takes an adult to compare two presented fractions, what types of mental representations does he/she form? With the existing equivocal evidence, it becomes difficult to make solid predictions. However, it may be possible to bridge these two seemingly disparate findings in the literature. Cohen Kadosh and Walsh (2009) have recently hypothesized a dual-process model of numerical representations. In this model, there is an initial, automatic representation that is directly tied to the surface format of the presented number. Later, there is a refinement of the automatic representation that is influenced by intentionality, resources, task demands, etc. It may be the case that the initial, automatic representation formed is directly tied to the components of the fraction, whereas the more refined representation uses magnitude information. We directly test this hypothesis in the present study. Critically, we use a hand-tracking paradigm (Spivey, Grosjean, & Knoblich, 2005; Freeman & Ambady, 2010) to gain insight into the online formation of fraction representations.

In the present study, we asked participants to quickly decide whether a presented fraction was smaller or larger than ½. During the task, we collected the streaming (x,y) coordinates of a computer mouse as they clicked on the correct response. By directly manipulating fraction magnitude and component size, we tested the selective influence of both factors on the trajectories of participants' hands as they made their decisions; this allows an unprecedented window into the formation of their mental representations (Freeman, Dale, & Farmer, 2011). If participants are indeed forming immediate representations based on components alone, then component size should have more of an influence on average hand trajectories than fraction magnitude. If, on the other hand, participants' immediate representations are based on magnitude, then we should see little difference in the trajectories of fractions with large components and those of equivalent magnitude with small components.

Method

Participants

26 undergraduate students (14 female, mean age 23.1 years) participated in exchange for partial course credit.

Stimuli

The fractions presented to participants were chosen by crossing the factors of fraction magnitude (smaller than ½, larger than ½) and component size (larger than 5, smaller than 5). Within each of these four cells, we chose two fractions (see Table 1).

Procedure

Participants were told that for each trial, they would be asked to quickly and accurately choose whether the presented fraction was greater or smaller than the target fraction ½. At the

Table 1

Fraction Stimuli, presented as a function of Magnitude and Component Size

	Component size	
Fraction magnitude	Smaller than 5	Larger than 5
Smaller than 1/2	1/4, 1/3	2/8[a], 3/9[a]
Larger than 1/2	2/3, 3/4	6/9, 6/8

Note: [a] To preserve magnitude in this condition, only denominators are larger than 5.

beginning of each trial, a button labeled START appeared at the bottom center of the screen, along with the two response labels SMALLER and LARGER presented in the upper left and right corners of the screen. Participants completed one block of trials with the labels ordered SMALLER -- LARGER from left to right, and the other block had labels ordered LARGER – SMALLER from left to right. The order of these blocks was counterbalanced across participants.

After participants clicked the start button, one of the 8 stimulus fractions randomly appeared in the center of the screen. Participants were then required to quickly click on the response label appropriately designating whether the presented fraction was larger or smaller than ½. During these responses, we recorded the streaming (x,y)-coordinates of the participants' computer mouse movements (with a sampling rate of approximately 70 Hz). To present stimuli and record mouse trajectories during responses, we used the MouseTracker software package (Freeman & Ambady, 2010). In order to guarantee that mouse trajectories reflected online processing, we instructed participants to begin moving their computer mouse as quickly as possible. In the event that the mouse initiation time exceeded 250 ms, a message appeared on the screen after the participant's response, instructing them to start moving earlier on future trials, even if they were not completely sure of their response. In total, each participant completed 120 trials (60 in each response label ordering).

Results and Discussion

To prepare the raw mouse trajectory data for analysis, we performed an initial preprocessing with the MouseTracker software package (Freeman & Ambady, 2010). All mouse trajectories were rescaled into a standard coordinate space (x-coordinate range: -1 to 1;

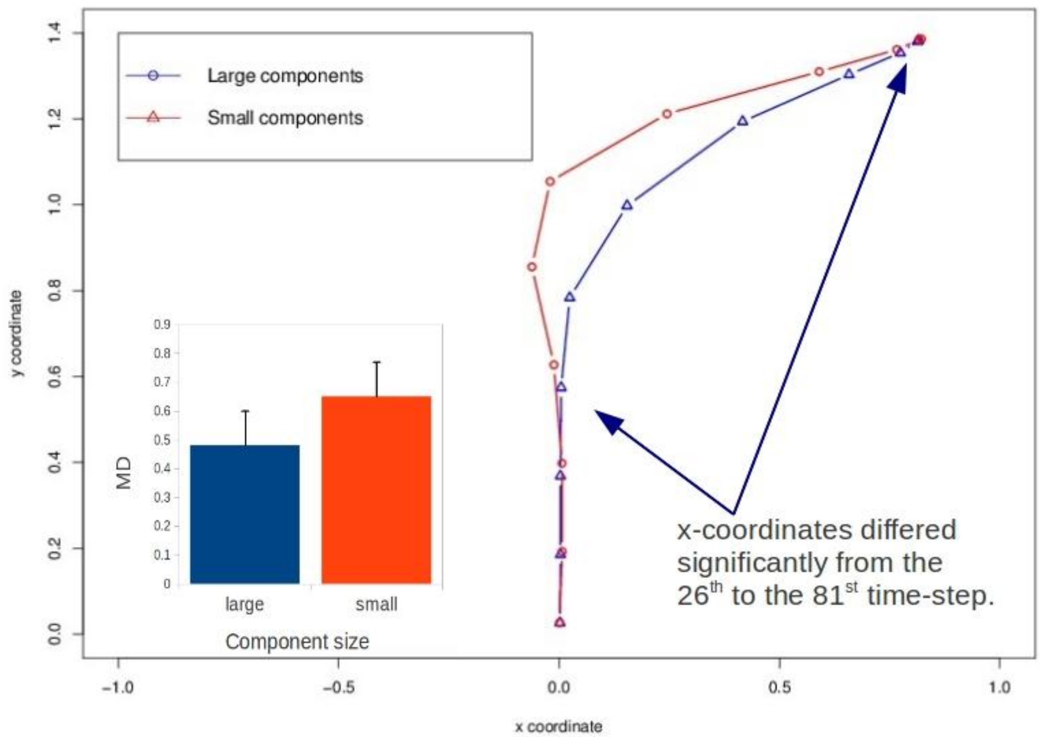

Figure 1: Mean trajectories and MD values for large fractions as a function of component size.

y-coordinate range: 0 to 1.5). In addition, to remove the confound of varying response times, all raw trajectories were normalized (via linear interpolation) to consist of 101 time steps. This step was critical in order to allow us to average across trials with differing time durations. As an index of trajectory complexity, we measured the degree to which the incorrect response alternative influenced participants' decisions by computing the maximum deviation (MD): the largest perpendicular deviation between the actual trajectory and the ideal response trajectory, represented by a straight line from the trajectory's starting point and the correct response (see Figure 1). Subsequent analyses were performed using linear mixed effects modeling (Pinheiro & Bates, 2000; Bates, Maechler, & Bolker, 2011) with the R statistical package (R Development Core Team, 2011).

Fraction performance

Participants were very quick and accurate to judge whether presented fractions were smaller or larger than 1/2. Across 3,120 trials, only 133 were in error (4.3% error rate). Overall, the mean reaction time of the correct trials was 1247 ms (*SD* = 546 ms). Outlier screening was initially performed; we rejected correct trials if the respective RT exceeded 3 SD away from the mean. 58 trials were discarded (1.9%). To analyze the influence of component size on the decision process when participants made their responses, we separately considered those fractions that were larger than ½ and those that were smaller than ½.

Large fractions

All fractions analyzed herein were larger than 1/2; hence, the correct response for all stimuli was LARGER. For ease of visualization and interpretation of these mouse trajectories, we remapped all trajectories to the right side of the display. Then we computed a mean trajectory for fractions with small components (2/3, 3/4) and fractions with large components (5/6, 7/8). As can be seen in Figure 1, trajectories for fractions with small components exhibit a great deal of continuous attraction toward the incorrect alternative (SMALLER), compared with fractions having large components. This effect was statistically significant; as indexed by maximum deviation (MD), trajectories for fractions with small components (fitted MD = 0.50) were significantly attracted toward the answer SMALLER relative to fractions with large components (fitted MD = 0.18), $t = 12.28$, $p < 0.0001$.

Small fractions

Similar to the previous analysis, all fractions analyzed herein were smaller than ½; hence, the correct response for all stimuli was SMALLER. This time, we remapped all trajectories to the left side of the display. Then we computed a mean trajectory for fractions with small components (1/3, 1/4) and fractions with large components (1/6, 1/8). As indicated in Figure 2, there was a large difference between the trajectories for fractions with small components and fractions with large components. Again, this effect was statistically significant; as indexed by maximum deviation (MD), trajectories for fractions with large components (fitted MD = 0.45) were significantly attracted toward the answer SMALLER relative to fractions with large components (fitted MD = 0.19), $t = 18.04$, $p < 0.0001$.

The role of magnitude?

Previous studies (e.g., Meert, Gregoire, & Noel, 2009; Faulkenberry & Pierce, 2011) have shown that people tend to process the overall magnitude of fractions during fraction comparison tasks. Typically, this effect is quantified by regressing reaction times with the distance between fractions to be compared. The presence of a *negative slope*, known as the numerical distance

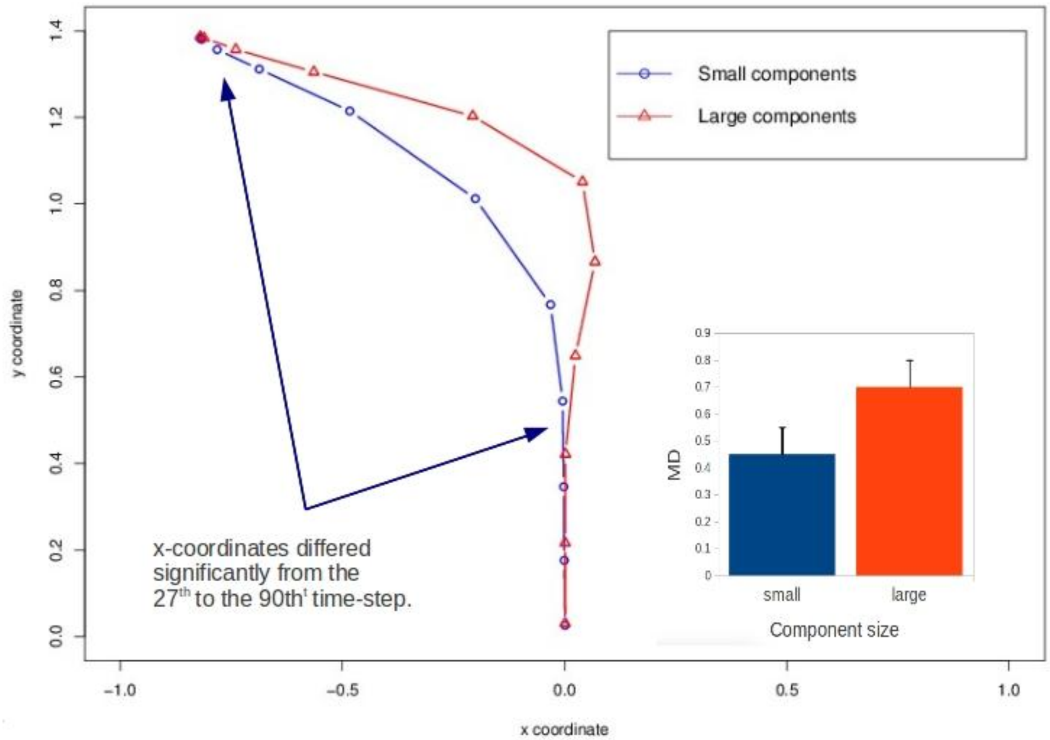

Figure 2: Mean trajectories and MD values for small fractions as a function of component size.

effect (NDE; Moyer & Landauer, 1967), is typically taken as evidence of participants' magnitude-based representations of numbers.

To assess whether participants in the present study attended to the overall magnitude of the presented fractions, we computed a linear mixed-effects model (Pinheiro & Bates, 2000) in R using the `lmer` package (Bates, Maechler, & Bolker, 2011). At the first stage, we computed a mixed-effects model with RT as a dependent measure, distance as a fixed effect, and participant as a random effect. The presence of the random effect term allows the intercept to vary for each participant while assessing a fixed slope, or effect, for distance across all participants. Critically, this model was fitted with a slope estimate for the distance fixed effect of -374.8 (t = 3.27). As this modeling is done within a Bayesian framework, "significance" is assessed via other means. One method is to compute a Bayesian analogue of a confidence interval for the slope (the 95% HPD, or highest posterior density) using 10,000 bootstrap samples. We found the 95% HPD to be (-597.4, -141.9). Both of these pieces of evidence indicate that numerical distance is indeed a significant predictor of reaction times. In other words, participants seem to be attending to the numerical value of the presented fractions, even though their mouse trajectories indicate that their decisions are quite influenced by the size of the components in the fractions.

The present data supports the dual-process model of Cohen Kadosh and Walsh (2009), whereby participants' initial, automatic representation that is directly tied to the surface format of the presented fraction. This was evidenced by the consistent effect that component size had on participants hand trajectories in the fraction comparison task: when component size was inconsistent with the overall magnitude of the fraction (i.e., large components, but small overall magnitude), participants hands tended to drift away toward the incorrect answer before eventually settling in picking the correct one. Also predicted by the dual process model is a later refinement of the automatic representation that is influenced by intentionality, resources, task demands, etc. In the present experiment, we hypothesized that this would be where the magnitude representation would come into play. Indeed, the current data supports this; through mixed effects modeling, we were able to find a consistent negative slope when regressing reaction times on distance, indicating that fractions farther from ½ took less time to respond to than did fractions that were close to ½. This is a classic marker of magnitude-based representations (Moyer & Landauer, 1967).

General Discussion

The present research may provide a bridge between some seemingly contradictory findings in recent research on fraction representations. We found that adults form fraction representations that attend to both the components *and* the magnitude of a fraction. While this may seem obvious, the magnitude part of these results is a bit trickier to resolve. In the present task (deciding if a fraction is greater than or less than ½), there is no reason, *a priori*, for someone to think about "how big" the fraction is. Indeed, the task could easily be taught to a child using without having to have a solid knowledge of fractions. However, the present data indicates that magnitude does indeed play a part in our mental processing of fractions. This has important ramifications for teaching: since magnitude is a critical part of successful adult representations of fractions, it is important that children gain a knowledge of fractions not only from a symbolic, component-driven view, but also their underlying numerical values.

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LEARNING MATHEMATICS IN THE 21ST CENTURY: HIGH SCHOOL STUDENTS' INTERACTIONS WHILE LEARNING MATHEMATICS ONLINE

Cherie Ichinose
California State University Fullerton
cichinose@fullerton.edu

When choosing online learning, educators and administrators are no longer limited in providing the most effective learning environment for their students. This study provides evidence of the power of online interactions when learning mathematics. This study showed algebra and pre-algebra students benefited most from the synchronous, live interactions through web-conferencing. Students were able to ask their questions in real-time, thereby fostering deeper relationships with their teacher. More advanced students preferred the asynchronous (any-time) environment for learning mathematics as shown through the increased collaborative participation through the online discussion boards.

In the new age of education, parents are able to explore different learning environments for their children. The North American Council for Online Learning (NOCAL) reported that, as of the end of 2006, there were 38 state-led online learning programs educating approximately 65,000 children (Watson, 2007). A more recent report published by the Sloan Consortium estimated the number of K-12 students engaged in some kind of online course in 2007-2008 at over one million (Picciano & Seaman, 2009).

NOCAL reports one of the top ten myths with online learning is the perceived lack of interaction that occurs (2007). Students and teachers typically report an increased level of interaction with the content and rich one-on-one interactions with each other (Learning, 2007). In *Learning in the 21st Century: 2010 Trends update*, students reported they received more attention from their teachers and were more comfortable asking questions online than in the traditional setting (Tomorrow, 2010). These separate physical interactions take on different modalities, however, and can be divided into two categories: synchronous and asynchronous.

Theoretical Framework

With most online courses, students can log into their courses at any time (*asynchronously*) and engage with the content through a learning management system. Asynchronous interactions include engaging with online learning modules and working with flash or multi-media activities. Students no longer passively read through mathematical content, but interact with it by linking visual and symbolic mathematical representations through guided simulations that lead to step-by-step support of mathematical processes and immediate feedback (Snelson, 2002; Suh &

Moyer, 2007). At his or her own pace, the student can work though the activity and then receive immediate feedback through the computer.

A threaded discussion board is also an example of an asynchronous interaction, as is e-mail. Threaded discussion boards are organized systematic learning tools that allow students to post questions anytime time and receive feedback directly from either the teacher or other students (Holden & Westfal, 2010). When the initial content is delivered by the computer, a teacher is able to spend more time focused on discussion by providing students the opportunity to foster a deeper level of quality mathematical understanding through discourse (Smith et al., 2003; Warschauer, 1997). Collectively, teachers and students have the opportunity to share their ideas, elaborate on their thought process, and compare their ideas with previous statements or work (Simonsen & Banfield, 2006). Smith (et.al.) charges that, in this new milieu, there is a higher expectation of teachers' accuracy; specifically, that "their standing may be only as good as their last posting" (p.53). The asynchronous nature of the medium allows the teacher the time for necessary reflection to compose appropriate responses to individual student questions as opposed to traditional settings that require answers on the fly or apparent delayed responses, which can harm teacher credibility (Smith et al., 2003).

Unlike the anytime model that asynchronous activities provide, *synchronous* activities occur in real time and are often described as similar to those in a standard chalk-talk classroom (Wahlstrom et al., 2003). As with a traditional brick-and-mortar setting, a group of students who log in at a specified time can learn all together, but without a shared physical presence (National Forum on Educational Statistics, 2006). Studies reported that students felt a strong social presence with their teacher, thereby fostering a strong sense of participation in the class (Anderson & Kuskis, 2007; Watson & Ryan, 2006).

The balance between such interactions is key. Students can solely interact asynchronously through the online content but as Holden and Westfal suggest, this may be more appropriate for activities involving drill & practice (2010). However, Anderson suggests synchronous environments are "particularly rich and encourage the development of social skills, collaborative learning, and the development of personal relationships amongst participants as components of the learning process" (2003, p. 9). Moore charges that true potential of online stems from the use of each interaction. That is, the program must match the appropriate interactions according to subject areas as well learning styles and developments (Moore, 1989).

The interactions that students experience between the content, teacher, and other students should not be studied in isolation of one another, nor is one greater than the other. Within the omni-media environment, the combinations of such interactions are possible. Because research in online learning relatively new, few studies only compare effectiveness of learning mathematics online when compare to its brick-and-mortar counterpart. A major weakness in these studies is the singular focus on one or another component part rather than the interplay between types of interactions (Anderson & Kuskis, 2007; Moore, 2007, 1989). The purpose of this study was to investigate the relationship between the actual use and perceived effectiveness of synchronous and asynchronous modes of delivery in learning mathematics.

Methodology

For the purpose of this study, a sample was drawn from 2051 high school students taking pre-algebra, algebra, and geometry courses online from one of nine virtual academies in California. The 458 self-selected students participated in an online mathematics course, with optional discussions through online web conferencing and threaded discussion boards. The curriculum was delivered through a learning management system (LMS) that that supports all instructional components of a traditional brick-and-mortar course. That is, students freely log into one place and interact with computer-assisted instruction, animation and simulations, java applets, streaming audio and video, live up to date grade-book, online practice, and assessment activities. By the end of the semester students were required to complete online quizzes and exams to satisfy the course requirements.

An online survey was administered to measure student preference, use, and confidence level in learning mathematics when learning is dependent on the online content. Additionally, the same was asked regarding threaded discussion boards, and synchronous interactions.

The development and use of frequencies, descriptive statistics in this article are based data collected as part of a larger study on students' interactions while learning mathematics online. Comparisons were made between the subgroups by performing a series of two-sample independent t-tests. In each case, a Levene's Test was performed to test the assumption of equal variances. The discussion that follows has been abbreviated; only parts of the larger study related to course behaviors (geometry, algebra pre-algebra) will be discussed. Further frequency distributions revealed which types of online student interactions were used most and their perceptions of said interactions. The average time spent on each asynchronous activity was logged by the system and then retrieved for analysis. Synchronous engagement was measured by the average number of minutes students spent attending such sessions.

Findings

Students were asked which interaction they felt was most useful for learning mathematics online. Of the students who answered the question, students felt that the online content was most useful for learning mathematics (44%), followed closely by synchronous sessions (40%). Table 1 shows a slight majority of geometry students favored the online content interaction (55.1%) for learning mathematics.

Table 1

Interaction Most Useful for Learning Mathematics

	Reading the Online Content	Attending or Watching Recorded Sessions	Threaded Discussions	E-mailing or Calling Teacher
Pre-Algebra	7 (25.9)	18 (66.7)	1 (3.7)	1 (3.7)
Algebra	65 (43.9)	68 (45.9)	4 (2.7)	11 (7.4)
Geometry	124 (55.1)	94 (41.8)		7 (3.1)

Note: Valid frequency in ()

Table 2 shows geometry students spent, on average, 31.02 minutes interacting with each online lesson, compared to pre-algebra and algebra students (20.62 and 23.97, respectively). That is, geometry students spent statistically significantly ($t = -3.460$, $df = 307.93$, $p = .001$) more time interacting than both the pre-algebra and algebra students (10.82 and 7.05 more minutes, respectively).

Table 2

Interaction Times in Minutes

	Sample	Pre-Algebra	Algebra	Geometry
Reading the Online Content	27.6	20.6	23.97	31.02
Attending Sessions	46.1	64.7	48.5	42.1
Threaded discussions	8.4	1.1	1.1	14.5

Based on the student responses, threaded discussion boards were useful when learning mathematics; however, when asked which interaction variable students preferred only a small percentage indicated the use of threaded discussion (less than 4%). Students spent an average of 8.43 minutes engaging with each post in their respective classes. The differences in discussion board usage between courses varied. Geometry students spent an average of 13 minutes more on each threaded discussion than the pre-algebra and algebra subgroups. ($t = -4.878$, $df = 169.10$, $p<.001$). Further, geometry students posted approximately twice as many posts than did pre-algebra and algebra students.

Table 1 shows the majority of algebra (pre-algebra and algebra) students strongly indicated that synchronous sessions were more useful in learning mathematics (66.7% and 45.9% respectively). On average, students spent 46.13 minutes attending each synchronous session. Combining the pre-algebra and algebra average synchronous session time yields a mean of 51.1 minutes, approximately 9 minutes more time per session than geometry students ($t = 4.669$, $df = 306$, $p<0.001$). One notable finding, although not significant, is that pre-algebra students, on average, attended approximately 3 more sessions than algebra and geometry students. Additionally interesting, geometry students reported with higher levels of agreement that they engaged by watching *recorded synchronous sessions* when compared to pre-algebra and algebra.

Conclusion

This study explored the interactions that occur when learning mathematics online. The primary purpose of this study was to investigate the relationship between the actual use and perceived effectiveness of synchronous and asynchronous online interactions when learning mathematics. The significance of this study was two-fold. First, because online education is relatively new, there are few studies examining effectiveness of learning mathematics online. Second, there is a deficiency of studies focused on high school age students taking online mathematics courses.

There was consistency between course subgroups and preferred learning modality. Algebra students (pre-algebra included) favored the *synchronous* interactions where geometry students favored the *asynchronous* interactions: threaded discussion, online content, and the use of pre-recorded synchronous sessions.

Overall this study showed students felt that online asynchronous interactions were effective for learning mathematics. As a stand-alone modality, this reveals the importance of a strong curriculum. The nature of the online content itself, without direct influence from the teacher, must be strong on its own.

When asked the overall effectiveness of threaded discussion boards for learning mathematics, a small number of students indicated that threaded discussion boards were effective. It is important to note that threaded discussions were not a requirement of the course. Geometry students' motivation to use threaded discussion as a means to get help was not influenced by an external influence of a graded course requirement; this suggest a need, lends itself for more research.

An examination of synchronous interactions revealed that students felt that synchronous activities were an effective way of learning mathematics and half reported that this modality was their choice interaction. Again, for this study, participation in such synchronous activities was not a requirement of the course. For pre-algebra and algebra students, their preferred interaction for learning mathematics online was through synchronous activities. This was explained by the higher levels of participation between the pre-algebra/algebra and geometry subgroups. However, there is little research on the effectiveness of synchronous activities. Only one study cited that synchronous activities produced no notable differences in achievement (Lou et al., 2006). This indicates the need for more research.

There were several *limitations* in this study. First, the survey analysis depended on the information provided by the students themselves. It also depended on the students' eagerness to participate in the study and on their truthfulness. Next, there are limitations with the interactions logs. There is no way to account for the actual use other than the time recorded. For example, if students merely log into the content and walked away from the computer the time is logged by the computer as active. There no way to differentiate this type of logged activity from a student who is actively reading the content. However, the same can be argued with a student sitting in a classroom who does not pay attention to the teacher. Further, one could argue that one course; geometry may have more interaction opportunities than say the algebra courses, therefore may explain the elevated participation time online.

The collection of synchronous interactions is not without flaw. The synchronous collection of data was based on student login information. If a student did not properly log in, their attendance was not recorded. Future upgrades to the system will include integration with students' identifiable information. Additionally, use of threaded discussion board and synchronous activities was not currently required by all instructors; this may have influenced students' perceived usefulness and actual interaction.

When teachers and administrators are making decisions regarding the online option for their students, considerations need to be made with respect to synchronous and asynchronous interactions. This study revealed that the importance of a strong interactive mathematics curriculum is vital for student achievement. Students need the opportunity to not only passively read a rich mathematical content, but also be an active participant engaging in simulations that allows for the transfer of mathematical knowledge to mathematical practice and application.

This study also revealed that the use of threaded discussion boards should be done with caution. Threaded discussion boards can be a powerful tool if used properly. Teachers and students can engage in rich mathematical dialogue within the comforts of their own time and learning environments. Participants can thoroughly research questions and answers before engaging and responding, thereby strengthen the mathematical dialogue.

Finally this study showed the power of synchronous interactions. Online course are often used to remediate mathematics skills to the most struggling students. Often students are given access to software in to supplement the current learning environment. However students, especially low-level students should not be learning online in complete isolation. As seen from this study, algebra and pre-algebra benefited from the synchronous interactions. Students were able to directly ask their questions in real-time, there by fostering deeper relationships with their teacher.

There is limited in research pertaining to online learning, specifically learning mathematics. Future research must focus not only on higher education, but more importantly on the k-12 learning population. This study showed the importance of interactions in general, but future studies are necessarily to find the power of such interactions. Future investigations must study each interaction, and combination of, as they influence not only the perception of learning mathematics, but also overall mathematical achievement.

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METAPHORS AS A MEDIUM FOR HERMENEUTIC LISTENING FOR TEACHERS

Sean Yee
California State University, Fullerton
Syee@Fullerton.Edu

Metaphors span language in many forms. Mathematics educators regularly use metaphors to relate one mathematical concept in terms of another. Embedded within these concepts are a set of shared experiences that educators use to communicate new ideas to students. However, students also use metaphors to convey their experiences, perceptions, and understanding. This paper reports on a large mixed-methods study that discovered a coherent set of metaphors that high school students and teachers associated with mathematical problem solving. Moreover, this methodology helps teachers learn how to listen hermeneutically for student experiences. This methodology is being developed for teacher professional development.

This study began with a pilot study that identified that high school students are capable of using complex metaphors (Lakoff, 1993) to describe their perceptions of mathematical problem solving while solving problems. This allowed for a larger study to determine the existence of a coherent system of conceptual metaphors (Kovecses & Benczes, 2010) related to problem solving used between teachers and their students. The results of this larger study are presented in this paper. As with all good studies, the results have raised more questions than answers. One such question was, "How are teachers listening to the students' metaphors/experiences?" Surprisingly, the application of this study's novel methodology (Conceptual Metaphor Theory, CMT) has demonstrated a means to listening in the classroom. Specifically, the researcher identified that CMT analysis can move teachers from a constructivist paradigm of interpretive listening to a hermeneutic model of listening for conceptual understanding (Davis, 1997).

Related Literature

The Greek word, *metaphora*, means to transfer or carry over (Presmeg, 1997). Metaphors are currently defined as denoting one figure of speech as another (Merriam-Webster, 2011). However, their use spans far beyond speech. Over the last 30 years, metaphors have been identified in language not only for communication, but for cognition and education (Sfard, 1997; Lakoff & Johnson, 1980; Ortony, 1993). This advancement can be implemented in the classroom to help improve student learning through teacher listening.

Students bring significant experiences to the classroom that happen outside of school, yet teachers lack a means to access such experiences under a sociocultural theory (Cobb, 2007). A distributed cognition theory (Cobb, 2007) allows such experiences to embed themselves on the curriculum locally, within individual classrooms. Distributed cognition allows students' external experiences to aid their learning within the classroom, if the teacher is willing to let the student express re-presentations(Glasersfeld, 1991) of those experiences. Metaphors are natural expressions of shared experience. Often within education, the purpose of metaphors are to share one person's experiences through common experiences (or experiences believed to be common) (Ortony, 1993).

A conceptual metaphor is a mapping, an identification of the experience to be expressed (the target domain) and the experience to be shared (the source domain) (Lakoff & Johnson, 1980). For example, a student said during my interviews, "to solve it for me, it meant that I had to find it somehow." Thus the student used the conceptual metaphor: PROBLEM SOLVING IS SEARCHING. Thus the student was sharing how their understanding of problem solving (target domain) is perceived as searching (source domain) (Kövecses & Benczes, 2010). The student communicates this because they believe the researcher has shared the experience of searching and so relates this perception to the researcher. Due to the specifics of each domain, it is important to note that this is a unilateral relation: TARGET DOMAIN →SOURCE DOMAIN.

The method of identifying conceptual metaphors and interpreting their purpose and meaning was first accomplished by Lakoff and Johnson (1980). Lakoff and Nunez (2000) expanded on this idea to create Conceptual Metaphor Theory (CMT) and Danesi (2007) applied it to middle school mathematics teachers. Kovecses and Benczes (2010) offered the first technique of looking for coherency concretely using CMT analysis in linguistics by identifying a conceptual metaphorical system. However, these methods were always narrative, interpretive, and qualitative methodologies. Moreover, there was a lacking of quantitative analysis from a post-positivist perspective allowing probability of metaphors to solidify the validity of a given metaphorical system. This is where my study takes shape and has generated a unique methodology that is replicable, but more importantly, a practical means to aid teachers in listening to their students meaning.

Participants

Participants for this study included students from multiple honors geometry classes and all honors geometry teachers in a suburban high school. Honors geometry was chosen due to the study being volunteer-based and due to the proclivity for proof to have students express their problem solving ability (Lakatos, 1976). The students and teachers both met with the research for 10-15 minute interviews about specific problems from one of three recent common assessments designed and determined by the teachers. The researcher chose problems that would require students and teachers to express their perceptions of problem solving. The research would first interview the honors geometry teachers and have them explain how they would expect students to approach the problem. From the population of honors geometry students, the researcher methodically randomized volunteers of the study by their teacher's assessment of their performance. The study included 22 independent student interviews and 6 independent teacher interviews.

Novel Methodology

The uniqueness of the methodology for this study stems from the two-step process of identifying the metaphors via a modified Interpretive Phenomenological Inquiry (IPA) (Eatough & Smith, 2008) that uses CMT analysis to identify conceptual metaphors, followed by a quantitative analysis of the frequency and popularity of conceptual metaphors to identify a coherent subset that are commonly used for the classroom. In the former step, IPA is more complicated than classical phenomenological studies because coding the data is not performed by identifying identical words used by the participants. Conceptual metaphors are identified by relationships and association, thus two students could use the same source domain for the target domain of mathematical problem solving without using similar words:

- "I was thinking it would be the easiest way."
- "I just go right into it because I know how to solve these."

Both of these literal metaphors from participants of the study create the same conceptual metaphor, PROBLEM SOLVING IS A JOURNEY, without using similar words. The coding occurs with the shared experience, the source domain. These source domains were not previously categorized as this was a phenomenological design. Often, the coding requires corroboration by context and student elaboration. Thus the interviews with students were semi-structured so as to allow the researcher to ask the participant to elaborate on their understanding.

After completing the qualitative stage via CMT analysis, this study focused on understanding students' metaphors for mathematical problem solving, and thus the source domains associated with the target domain of problem solving. Results were tallied quantitatively within two dimensions: popularity and frequency. Frequency tallied all of the source domains and broke down, by percentage, which source domains were most frequented by students and teachers.

Popularity tallied whether or not a student or teacher used a given metaphor. The need for both quantitative analyses was necessary to verify that one students' abundant use of one source domain did not bias the results. For example, what if a student used the source domain of JOURNEY associated with the target domain PROBLEM SOLVING 100 times in the interview while another student frequented the term only once? This may skew the data if frequency alone was studied. Thus popularity was a means to verify internal validity for this methodology.

Results

The CMT analysis discovered some surprising relationships (and lack thereof) between teacher and student thinking. Table 1 is a list of source domains used by students, and source domains used by teachers related to problem solving.

Table 1

Teacher and Student Source Domains Associated with Mathematical Problem Solving

Student (22)	ABILITIES, ACQUISITION, APPROXIMATING, BUILDING, CALCULATING, COMPARING, CONDITIONAL, CONTEST, CONVINCING, DISCOVERY, EXPERIMENTING, IMAGINING, JOURNEY, PARTITIONING, PROCESS, PROVING, REVIEWING, SEARCHING, THINKING, VISUALIZATION, VOCALIZATION, WAR
Teacher (22)	ACQUISITION, BUILDING, CHANGE OF STATES, CONFLICT, DISCOVERY, DOING BUSINESS, FAMILIARITY, GENERALIZING, HABITS, IMAGINING, JOURNEY, PARTITIONING, PROCESS, REVIEWING, RACE, RULES, SEARCHING, SETS OF SKILLS, TOOLBOX, VISUALIZATION, UP, WAR

There is a significant overlap between student and teacher source domains. Students used a total of 22 source domains while teachers used 22 source domains. There were significantly more student interviews than teacher interviews, so it is natural that the diversity in metaphors used were in the students' favor.

There were source domains used by the teacher which were lacking in the student's source domains. For example, one teacher used the conceptual metaphor of PROBLEM SOLVING IS DISSECTING.

> "They've been taking these shapes and breaking them up and dissecting them into various more familiar shapes…. because the triangle ends up being the right triangle, they'll look at that and dissect into those two triangles . . . We've talked about dissecting the problem. I think they will see this as being easier to dissect than it will be so surround the shape with a rectangle and subtract the triangle…"

Notice, there is no other experiential reference to dissection. The teacher never refers to cutting inside or slicing or any surgical operation language. This is a demonstration of what Max Black (1962) referred to as dead metaphors due to their lack of interaction with other aspects of the current experience and the rote use by the participant. Moreover, the source domain of DISSECTING was never used by any students (including his).

Additionally, different teachers used different source domains but to the same end. Teacher1 said "hopefully one of those things that they have physically done will spark their memory". Teacher2 states, "In my opinion that will be the first term that that jumps out at them." Both teachers are referring to recognizing/discovering information about the given hexagon. While one is referring to sparks and fire, the other is referring to "jumping out". Both experiences of ignition of fire and jumping are actions that are perceived as sudden. Both actions are a changing of states, jumping is moving from static into motion, ignition is moving from not burning to burning quickly. Specifically, both actions do not indicate how the change of states was achieved, but is rather difficult to understand such a change. This leads back to the word "realize" as, again, the notion of *how* a student is to be aware of this transformation (spark or jump) is purposefully not described by the teachers. Thus the only coherent aspect of problem solving that can be drawn from such complex, linking metaphors (Kövecses & Benczes, 2010) is that PROBLEM SOLVING IS A CHANGE OF STATES.

The quantitative results demonstrated a strong similarity between the most popular and most frequented source domains. Table 2 demonstrates these results:

Table 2

Comparison of Teacher and Student Source Domains for Problem Solving

Students' Most Frequented Source Domains	**Students' Most Popular Source Domains**	**Teachers' Most Frequented Source Domains**	**Teachers' Most Popular Source Domains**
26% JOURNEY	95% JOURNEY	29% JOURNEY	100% JOURNEY
18% SEARCHING	86% VISUALIZING	17% DISCOVERY	100% DISCOVERY
13% VISUALIZING	82% SEARCHING	11% BUILDING	83% SEARCHING
12% DISCOVERY	73% PROCESS	10% VISUALIZING	83% BUILDING
9% PROCESS	68% BUILDING	7% PARTITIONING	67% VISUALIZING
8% BUILDING	68% DISCOVERY	7% SEARCHING	50% PROCESS
8% PARTITIONING	55% PARTITIONING	6% PROCESS	33% PARTITIONING
			33% RACE
			33% SETS OF SKILLS

Amazingly, despite there being a few more popular source domains from the teachers, the top seven source domains remained the same in popularity and frequency for both teacher and student. These results demonstrate a coherent system of conceptual metaphors within mathematical problem solving in these high school honors geometry classrooms.

PROBLEM SOLVING IS A JOURNEY, SEARCHING, VISUALIZING, A PROCESS,

DISCOVERY, BUILDING, and PARTITIONING.

Other analyses were done with respect to the data, such as comparing the students' score with the frequency and popularity of the metaphors. Surprisingly, with every single student metaphor, no correlation was significant with student performance. Not a single metaphor suggested a positive correlation with a student's score on an assessment. This is remarkable and encouraging as many mathematics educators strive to demonstrate multiple representations (Pólya, 1954). Other analyses will be discussed during the presentation.

Conclusions and Future Studies

The fundamental result of this study used a mixed methods methodology where a phenomenological qualitative analysis (CMT) was used to identify source domains (i.e. shared experiences) that students and teachers associate with mathematical problem solving. This was then followed by a quantitative analysis to identify the more dominant metaphors amongst all students by frequency and popularity. These dominant metaphors surprisingly aligned among teachers and students generating a coherent metaphorical system (Kövecses & Benczes, 2010). It is vital that this research is not misinterpreted. This study has only shown existence of a coherent metaphorical system, not uniqueness. This data is not attempting to prescribe these source domains as the only source domains for problem solving in any classroom. Instead, this study validates the prescription for the methodology and the use of CMT analysis as a means to identify localized coherent conceptual metaphorical systems. This is why the theoretical framework is one of distributed cognition rather than sociocultural theory (Cobb, 2007).

This novel methodology is a means to identify, stimulate, generate, coordinate, and disseminate shared experiences that students bring to the classroom. As this study contained only six teacher interviews, a current study is underway to interview 30 preservice, practicing, and master teachers' perceptions of problem solving using this methodology. Under the NSF-funded Fullerton MT2 grant, there are current studies underway with pre-service and master teachers to continue to develop an understanding of the conceptual metaphors used by teachers. The claim is that identifying teacher metaphors directly aids in teacher meaningful listening. These results will be discussed at the conference.

If the vehicle for this methodology is conceptual metaphors, teacher listening fuels this vehicle. In Michael Gilbert's (2005) article, *An examination of listening effectiveness of educators: Performance and preference*, he identifies how different personalities in education have different indicators of effective listening. Gilbert identifies that teachers expect students to listen 65-90% in the traditional classroom, but how do mathematics teachers listen? The personality type for most mathematics teachers is the persistor; conscientious, dedicated, and observant believers in their craft. According to Gilbert, these personalities scored very low on effective listening. This is no surprise as Gilbert recognizes the limited training we give to educators on how to listen.

The significance to teacher listening can be perceived from the pervasive need for all teachers to see metaphors as an intimate form of communication at all levels of education (Petrie & Oshlag, 1993), to the specific understanding that metaphors directly imply an embodied mathematics that is not mind-free (Lakoff & Nunez, 2000). Davis (1997) discovered through a series of vignettes that mathematics teachers listen through "concept study" with three levels of listening: evaluative listening (assessment-based paradigm), interpretive listening (constructivist paradigm), and hermeneutic listening (conceptual participation paradigm). The third form of listening Davis suggests is what teachers should strive to develop, an immersed perspective of genuinely participating and sharing in the cognitive experiential development of the meaning behind the mathematical topic. In no way is this restricted to problem solving. Problem solving was my means to discovering this methodology, but not its only use.

What my research proposes is that CMT analysis allows for this participatory conceptualization by analyzing and interpreting the shared experiences of the students through their conceptual metaphors. This is not to suggest that teachers must follow my methodology in the classroom as this is ridiculously time and energy consuming. Instead, I suggest through practice and professional development with conceptual metaphor theory, teacher's ability to listen her will advance from interpretive listening to hermeneutic listening so that mathematics is contextualized for the student in terms of their experiences and not the teachers.

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ROUTINES OF PRACTICE FOR SUPPORTING MATHEMATICAL CONNECTIONS: EARLY ALGEBRA CONTEXT

Jessie Chitsanzo Store
Alma College
storejc@alma.edu

Teaching mathematics as a connected subject is very important. In fact, the Common Core State Standards for Mathematics are meant to serve this purpose. However, several studies show that the intended curriculum often differs from the enacted curriculum. To realize the goal of teaching mathematics as a connected subject then requires studies of pedagogical practices that connect different mathematical domains. This study discusses how elementary school teachers supported students' activities in connecting mathematical ideas within and across different contexts and how this practice supported students' algebraic reasoning.

Historically algebra has been viewed as manipulation of symbols and equation solving; therefore, algebra has normally been identified with symbolic thinking (Smith 2008). Early algebra, on the other hand, views algebra as a human activity that involves doing, thinking, and talking about mathematical ideas (Kaput, 2008). National Council for Teachers of Mathematics, (2000) defined early algebra as mathematical thinking that "emphasizes relationships among quantities, including functions, ways of representing mathematical relationships, and the analysis of change" (p. 37). In the classroom, early algebra activities include expressing explicit generalizations as a description of systematic variations or relationships in pattern finding activities (Mason, 2008).

Algebra is identified as a gatekeeper for students' future success in school (NCTM, 2000). Early algebra is included in elementary school curricular to enhance students' algebraic reasoning and help them to be better equipped to succeed in formal algebra classes (Rivera, 2006). Furthermore, early algebra is a tool for developing children's mathematical proficiency because it promotes conceptual understanding, procedural fluency, and positions students well for higher-level mathematics (Kaput, 1999; Blanton & Kaput, 2011; Carraher, Martinez, & Schliemann, 2008). Not only is developing students' proficiency in early algebra a necessity for their current success, it is also necessary for their future success. However, while these potential benefits of early algebra are widely reported in literature, there is scarcity of research on classroom activities that show how teachers may realize these benefits (Store, 2012).

Emphasis on early algebra has simultaneously grown with emphasis on making connections when teaching mathematics. Drawing on Hiebert and Carpenter (1992), mathematical connections are connections or relationships of mathematical ideas and representations from tasks within and across mathematical domains and curricular areas. Moreover, making connections is the core of mathematical understanding. As Hiebert and Carpenter explained:

> The mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and strength of its connections. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections (p. 67).

While potential benefits of early algebra and mathematical connections are widely reported in literature, there is scarcity of research on classroom activities that show how teachers may realize these benefits and on how early algebra is a tool for mathematical connections. The purpose of this study is to contribute to research on creating classroom contexts for supporting students' development of mathematical reasoning as recommended by Proulx and Berdnaz (2009). This paper addresses this research gap by reporting how elementary school teachers made mathematical connections in their early algebra classes, thereby supporting students' beginning understanding of algebra, and understanding of other mathematical domains.

Theoretical Perspective

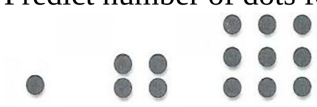
Greeno's (1998) situative learning perspective informs this study. This perspective views learning as individuals' construction of knowledge as they participate in communities of practice. Individuals' constructed knowledge is a tool for and a product of participation in communities. Since "individual mental structures certainly change as part of this learning" (Sawyer & Greeno, 2009; p.364), knowledge is transferrable from one community of practice to another. That is, students' impoverished understanding of algebraic ideas may reflect aspects of the communities in which they participated. "Situative approaches provide analyses focused on coordination of actions of individuals with each other and with material and informational systems" (Anderson, Greeno, Reder & Simon, 2000; p.12). A more specific methodological implication from this perspective is a focus on constraints and affordances of activity systems (Greeno, 2003). Affordances are aspects of an activity system that participants may use to reach their goals. Constraints structure the interaction between participants of a community and affordances. Based on this framework, the situative research question for the current study is: What are the affordances and constraints for supporting understanding of mathematical connections? This framework provides lens to look at how students may be supported to construct their knowledge to effectively participate in mathematical practices.

Methods

This longitudinal study is part of On Track-Learn Math research project that aims at supporting mathematical reasoning in elementary schools through after-school enrichment programs. Participating teachers attended professional development aimed at developing content and pedagogical knowledge. Participating students were in grades three through five at five elementary schools in the southeastern region of the United States. While students worked on different reasoning tasks, data from functional reasoning pattern finding tasks inform this study. Table 1 contains examples of the functional reasoning instructional tasks. Data (video, audio, field notes, student written artifacts) were collected from 30 one-hour lessons from each school in 2011.

Informed by the situative perspective, data analysis also focused on how teachers created opportunities for students to make connections and the reasoning co-constructed within such practices. Classroom activities were analyzed using constructivist grounded theory (Charmaz, 2011) data analysis procedures to identify instructional routines. Videos of classroom activities were transcribed and data entered into Nvivo. Line-by-line coding led into theme development and definition of categories of instructional routines. Follow up interviews were conducted with six teachers based on the identified instructional routines to understand teachers' meanings of and rationale for their practices. Multiple coders and member checking built in the trustworthiness of the analysis (Creswell, 2007).

Table 1
Instructional Tasks

Square Table Task	If one person sits on each side of a square in this pattern, how many people would sit around a train of 100 squares? Write your rule.  1 2 3 **Rule:** $y = 2x + 2$
Pentagon task	How many people would sit around a train of 100 pentagon tables. Write your rule for finding number of people that can sit around a train of *any* number of pentagon tables.  1 2 3 **Rule:** $y = 3x + 2$
Square number task	Predict number of dots for stages 5, 10, 100. Write your rule.  1 2 3 **Rule:** $y = x^2$
Function Machine	If I put 1 into the function machine, 15 comes out. If I put 2 into the function machine, 25 comes out. If I put 3 into the function machine, 35 comes out. What is the rule for this function machine? **Rule:** $y = 10x + 2$

Results

Four routines of practice that supported mathematical connections are reported. These practices are referred to as routines because they were teachers' regular instructional moves and became part of the classroom norms. These routines are connecting ideas across the curriculum, strategies, representations, and tasks.

Connecting ideas across the curriculum

During enactment of these tasks, several connections were made across the curriculum. This paper describes connections made between the aforementioned early algebra tasks with multiplications skills, exponents and properties of two-dimensional figures. The relationship between addition and multiplication was used as a tool to bridge third grade students' transition into multiplication. With the $2t + 2 = p$ as a rule for the square tables train task, teachers used $2t$

to explain that 2 times t is the same as $t + t$. When students were having problems multiplying numbers (e.g. 100 x 100 for 100th position of the square number tasks), teachers took up the opportunity to teach multiplication skills involving 10 with exponents of positive integers. Additionally, the square number task created opportunities for fifth grade students to teach ideas of exponents to fourth and third grade students. Classroom vignettes showing connections of ideas across the curriculum will be presented at the conference.

Rules for these early algebra activities were $2x + 2 = y$ for square table task, $3x + 2 = y$ for pentagon task, $x^2 = y$ for square number task, and $10x + 2 = y$ where x is the independent variable (input) and y is the dependent variable (output). As may be noted, these generalizations required some understanding of multiplication. By including a focus on developing other skills in the school mathematics curriculum, teachers positioned students to be able to make and understand algebraic generalizations while simultaneously supporting understanding of other mathematical areas.

Connecting multiple representations

Students were encouraged to use different representations. They often used physical manipulatives to model the tasks. For example, they used pattern blocks to explore patterns when working on train table tasks and square number tasks. They were encouraged to draw the geometric representation of the tasks (see Episode 1). According to the teachers' informal assessment, drawing geometric representations of the tasks helped students to identify what is varying and what is constant and supported students in making and understanding their generalizations.

Episode 1

> I think it (a pictorial representation) helps them to be able to visualize, for example with the tables that we have been having with people sitting around (table tasks) . . . it was just easier for them to visualize it when you know, you had the two on the side and you take the top number and the bottom number . . . but without that picture it may not have made sense to them.

Students were expected to collect data from their explorations and systematically organize them in input-output charts. As explained in Episode 1, students started exploring patterns with manipulatives and pictorial representations and moved to exploring patterns using input-output tables. Figure 1 is an example of a student's input-output table for the pentagon table task. Using t-charts made it easier for students to identify patterns because it “isolates pertinent information from everything else in the word problem.” Since students tended to look for patterns in one variable and developing recursive generalizations, teachers used t-charts to encourage students to look for patterns between the input and output variables and come up with algebraic equations. In Figure 3, a student used an input–output table to explore relationships between independent and dependent variable for the pentagon table task. The exploration led to a conclusion that for the pentagon task, multiplying the number of pentagons by three and adding two to the product gives the number of people that can sit around a train of pentagon tables.

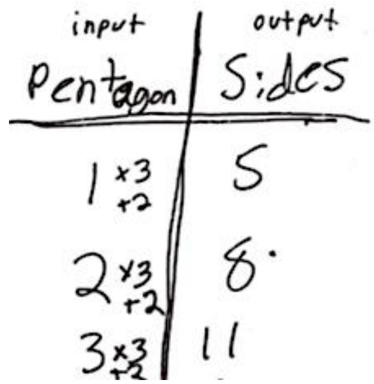
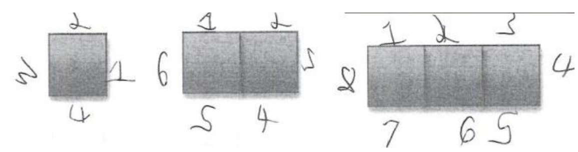
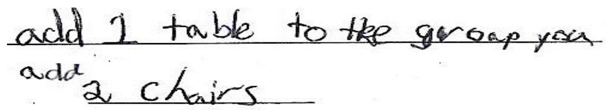
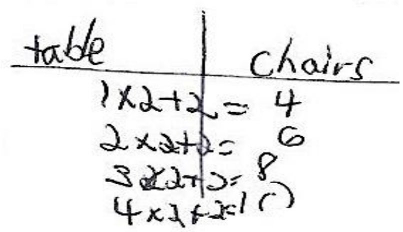
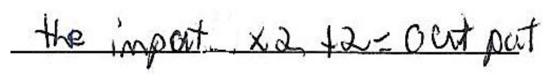
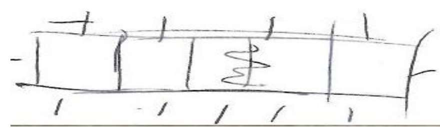


Figure 1. Student’s Input-Output Table for the Pentagon Table Task

Table 2 is a series of a student’s work that illustrates the teacher’s observation in Episode 1. After a student explored patterns with geometric representations of the square table task, he wrote a recursive rule. When he continued his explorations with a t-chart, he related the independent and dependent variable and wrote a correct explicit rule. This student went back to the geometric representation to explain that his explicit rule is valid because the number of seats would be equal to two times the length of the train table plus the two ends. This example shows student’s algebraic reasoning through connecting ideas from different representations.

Table 2

A Student's Series of Reasoning between Different Representations

Explorations with geometric representations	
Pattern observed after geometric explorations	
Explorations with t-chart	
Pattern observed after explorations with t-chart	
Geometric representation used to explain pattern observed from t-chart	

Connecting mathematical ideas from different student strategies

As reported earlier, whole class and small group discussions created opportunities for students to connect their reasoning to other students' reasoning. Teachers discussed how students' ideas were related to one another after using a variety of colors to write different student ideas on a whiteboard (see Figure 4). For example, in one class while working on the relationship between time and distance of a function machine task, one student referred to figure 15 and explained her strategy as "the number you have in minutes is going to be the same number in feet with a five at the end." To connect this strategy to the one whereby students multiplied minutes by 10 and then added 5 to get number of feet, the teacher asked, "Why is this generalization of writing a 5 in front working? Why is it giving the same number as multiplying by 10 and adding 5?" These questions led to a discussion about place value that by writing a 5 in front of the number, the value for minutes gets promoted to a tens place, therefore, the strategies are mathematically the same.

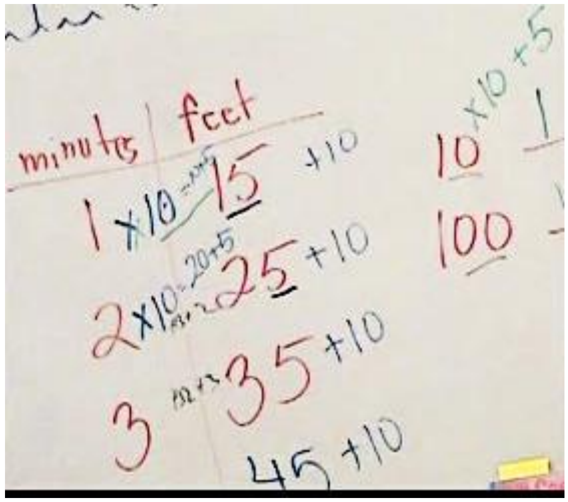
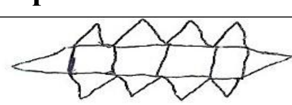
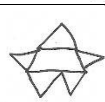


Figure 2. A Display of Student Strategies Used to Approach the Function Machine Task

Connecting Ideas from different Tasks

Table 3 is another example of how creating mathematical connections supported algebraic reasoning. In this example, a student made connections across tasks and different student strategies to reason about the pentagon table task. When this student was working on the square table task, she seemed to have observed a pattern between input and output values. She showed that for input of 1, the output is 1 + 3 and for 2 square tables, the output is 2 + 4. Although this reasoning could lead to a generalization of $x + (x + 2) = y$ for x tables, she did not take it further using input of 1 and 2 and could not explain it. From her drawing, it also seems she reasoned by considering available seats on each train. It appears she was on track to generating an explicit rule although she did not.

Table 3
Student's Connection of Mathematical Ideas

Task	Student's Representations and Generalizations
Square table task	 you add 1+3=4 then you add 2+4=6
Pentagon table task	 X3+2 You multipy the number of trains by 3 then add 2.

During whole class discussion, another student described his explicit rule as $2x + 2 = y$ (where x is number of tables and y is number of people) and explained that this rule works because each square table contributes 2 seats to the train and the train has 2 other seats on its ends. More than a month later, this student was exploring the pentagon table task. As seen from her representation, she used reasoning from previous task (square table task) to reason about the pentagon table task that $3x + 2 = y$ where x is the number of pentagon tables and y is the number of people that can sit around it.

Summary and Conclusion

Teachers situated learning for students to make connections of mathematical ideas from different contexts. These connections were between the tasks and (a) between students' strategies, (b) different representations, (c) different tasks, and (d) different ideas in the curriculum. Connections within one context (e.g., student strategies) also led to connecting mathematical ideas with other contexts (e.g., curriculum and tasks). The discussed example of a student's work in Table 3 illustrates the connections across different tasks and different strategies. Teachers positioned students to make these connections by asking them to think about how the tasks they were working on were similar to other tasks, asking students to identify underlying mathematical ideas of different strategies, and asking students to share their ideas during whole class discussions. Connecting mathematical ideas afforded students' algebraic thinking in making generalizations and validating those generalizations. The results show that the routines of practice reported in this study support students mathematical practices.

This study was conducted in an informal setting. Future research may explore routines of practice in both informal and formal settings to investigate how participants in mathematics learning communities can navigate affordances of each setting to further mathematical understanding. Future research may also focus on preparing mathematics teachers, who are able and willing to connect classroom mathematics activities with out-of-school activities in meaningful ways for students with diverse backgrounds.

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THE EVOLUTION OF STUDENT IDEAS: THE CASE OF MULTIPLICATION

Kris H. Green
St. John Fisher College
kgreen@sjfc.edu

Bernard P. Ricca
St. John Fisher College
bricca@sjfc.edu

The genesis of ideas, studied by Piaget and others, is important in understanding how students learn, which in turn, informs pedagogy. Using biological genetics as a metaphor for learning, we revisit previous work on multiplication, examine some teachers' explanations of multiplication and explore how these teachers use their explanations to generate new understandings of multiplication. This examination supports the use of the biological metaphor, and provides some insight into how to teach multiplication with attention to flexible thinking.

Taxonomies related to knowledge and learning abound (e.g., Anderson & Krathwohl, 2001; Kamii, Clark & Dominick, 1997). However, despite the large number of classifications of knowledge and the recent understandings of constructivism, relatively little is known about the dynamics of how these understandings are constructed by learners. This is likely due to the prevailing metaphors regarding learning (Thelen 1997, 2005). In this paper, we propose a metaphor for learning – that learning is similar to biological evolution in many respects – that enables a new look at the evolution of student ideas, benefitting researchers in student cognition and teachers planning curriculum and instruction.

Theoretical Framework

Genetic Metaphor for Learning

On one hand, a biologically-inspired approach is nothing new, as Piaget's genetic epistemology (Gallagher & Reid, 2002) followed his own training in zoology. Piaget's notions are insightful and influential when considering the *genesis* of knowledge, but do not take a *genetic* (in the biology sense) approach. Following Thelen (1997), we propose a new way to look at some of these same issues, and we believe that this new lens can aid teachers in ways that Piaget's ideas cannot. In particular, this new metaphor can more clearly guide instruction that incorporates existing student ideas, giving more guidance to constructivist approaches to teaching. Although discussing these individually is beyond the scope of this paper, this metaphor also unifies a number of approaches to cognition.

This piece is too short for a full explication of biological genetics, but the overview likely is known to most readers, and only some details are necessary for the current work. Roughly speaking, in sexual reproduction biological offspring both resemble their parents and differ from their parents in phenotype (observable characteristics) and genotype (genetic makeup), with phenotypes being the result of genotypes growing in an environment. In reproduction of this sort the genes of the offspring are usually some combination of two different genes from the parents; the dynamics of combining produces offspring whose genotypes are identical to their parents, or some mixture or hybridization, etc. In addition, mutations, such as “mistakes” in copying genes or the result of external factors “interfering” with reproduction, can occur. In biological evolution, phenotypes - and hence, genotypes - in a population are “selected” in a probabilistic manner through their fitness to the environment (e.g., “survival of the fittest”).

Learning viewed as a *genetic learning process (GLP)* parallels the biological ideas of phenotypes, genotypes, offspring, and selection. In this view, the ideas or concepts that a person holds are similar to the genes or alleles of biology. A GLP brings about new knowledge as an offspring of previous understanding through the replication of a previous idea or the combining of two (or more) existing ideas, or the mutation of a previous concept, and so on. The “fitness” of the offspring, however, is determined not so much by the environment as it is by the coherence or efficiency of the new scheme as viewed by the learner. Because this view is one of the dynamics of learning, and not merely a taxonomy of approaches students can take, it can provide additional insight into both teaching and learning.

Genotype vs. Phenotype in GLP

First, we note that there is an important difference between the genotype of knowledge and its phenotype. Let us consider two hypothetical students who answer the question “What is 12 × 23?” Each student (seemingly) follows the traditional algorithm, and gets the answer of 276:

$$\begin{array}{r} 23 \\ \underline{\times\ 12} \\ 46 \\ \underline{23} \\ 276 \end{array}$$

Despite seeing the same phenotypic (observable) multiplication behavior in each student, there are at least two quite different genotypic (but not easily observable) approaches to multiplication; often these differences become apparent only in later work. Verbal probing by the teacher might draw out differences between students (e.g., regarding place value or understanding of partial products) that could point to underlying genotypic variation.

For example, let us consider what happens to these students as they begin to study three digit multiplication. Suppose the teacher begins the study of three-digit multiplication with the problem 234 × 23. One possible genotype of the traditional two-digit multiplication algorithm is something like "Multiply the bottom right number by the top right number, then the bottom right number by the top left number, then the bottom left number by the top right number and finally the bottom left number by the top left number". This genotype will typically lead to confusion (or even despair) when confronted with a three digit multiplicand, for "left-right" may or may not admit a "center". (This depends upon how fully or deeply ingrained a learning allele is.) However, a student who has a more iterative gene (e.g., "multiply the right-most bottom column by all the columns on the top, then proceed to multiply the next bottom column to the left by all the columns on the top, and keep doing this until there are no more bottom columns") is likely to be bored by the new problem, for three digit multiplication *is nothing new*. Notice carefully that the possible reactions to the "new" material range from "no connection to the old material" to "nothing is new here". Certainly, many other reactions are possible, but even this example indicates that the genotype, not the phenotype, is important for learning.

A similar analysis could be used if this two-digit multiplication were used as a precursor to multiplying algebraic binomials (e.g., as a "warm-up problem" or to "activate prior knowledge"). In this case, a "left-right" student is likely, because of what amounts to a lack of distributive property ability, to write something like $(x + 2)(x + 2) = x^2 + 4$.

Based on these hypothetical students, it is apparent that the ability to solve a supposed "pre-requisite" problem is not sufficient to determine if the necessary pre-requisites are part of a student's cognitive make up; the ability to solve such a "pre-requisite" problem is not sufficient to indicate readiness for learning. Although this may not be surprising to many, two important points arise from this consideration: First, phenotype assessments, while easy to administer, do not result in a complete picture of student understanding. Second, curriculum should be focused not on what types of problems (phenotypes) should be solved by students but rather on what additional genes are needed to combine with a student's existing genotype. Although formative assessments are common, we claim that more meaningful formative assessment is not possible without considering the observation of genotypes. Thelen (1997, 2005) argued that metaphors both constrain and enable what we can see; the genetic metaphor calls for a different type of formative assessment. Further, considering the genetic processes that drive the dynamics of student learning calls for different approaches to instruction.

The first point is related to Vygotsky's notion of *zone of proximal development* (ZPD) (Vygotsky, 1978). Although Vygotsky did not put it in terms of genotype vs. phenotype, he did note that, although two students may have equal "mental age", their future performance could differ greatly; this is part of his argument for the existence of a ZPD. Hence, it is important that teachers assess something more than just phenotypes: Mere performance on problems is not necessarily a good indicator of future learning.

Second, planning for instruction is neither merely about careful presentation (a la the transmission model of teaching) nor merely about setting up an environment to explore. While the latter is closer to the mark, "setting up an environment" could be done solely by looking at phenotypes. Care must be taken, however, that the environment will present opportunity for changes in genotype, and this happens only when there are appropriate opportunities for genotype changes and the combination of existing genes.

The usual curriculum design process in schools - which does not take into account the actual understandings of the students, but rather posits a series of problems to be mastered - is not likely to consider the evolution of student genotypes. Successful mathematics curricula, such as those described in Fosnot & Dolk (2001) or Kamii (2000), while recognizing an overall structure of topics, also explicitly take into account student reasoning, and do so in ways that respond to student genotypes as well as phenotypes. Even though a full examination of curriculum development is beyond the scope of this paper, it is worth noting that a "well-constructed" sequence of problems is one which takes into account a genotypic view of student understanding.

Multiplication: A Classroom Example

Certified childhood teachers in a graduate course focusing on the understanding of basic mathematical operations were working with *alphabitia* (Bassarear, 2011) which essentially asks teachers to rediscover and explore basic operations in base-5. While studying multiplication, there were several opportunities to observe these teachers developing new (genotype) ideas as an offspring of prior ideas. During this process, artifacts from these teachers were collected and semi-structured interviews were performed with the teachers to gain insight into their work.

One (rather non-standard) demonstration of the combination of existing ideas occurred early on; the teacher's work is illustrated in Figure 1. The teachers were asked to justify that 3×3 was equal to 14, in base-5. Almost all the teachers drew a picture that was similar to Figure 1(A).

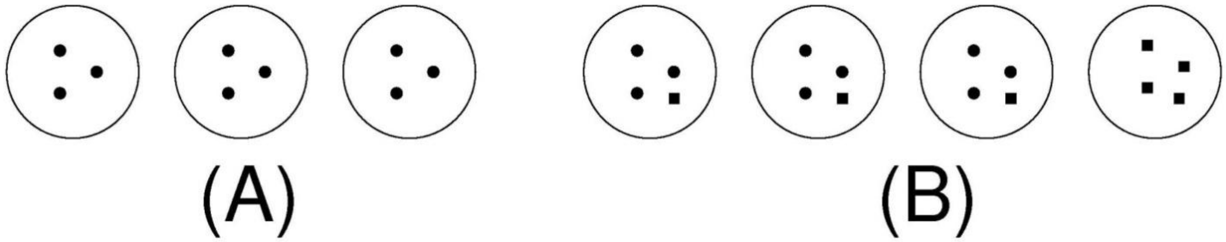

Figure 3. (A) Original drawing for 3 x 3. (B) Modified drawing for 4 x 4. Squares indicate additions to original drawing.

When asked to do the same thing for the problems 4 × 4, one teacher drew Figure 1(B), stating "I knew the answer to 3 × 3, and I knew how to count on. So I added one to every group [of 3], and I added a new group [of 4]." In this case, the parent genes were (a) her knowledge of 3 × 3 (or of groups of dots in general) and (b) her counting on strategy. When asked why she used this approach, the teacher responded that "They [the problems] were close together, so it was easy to do." In other words, the environment (problem) guided the notion of "fitness" of the idea. This new approach was used later when this student took a geometric approach to this problem:

Working in base-5, consider the sequence P_0, P_1, P_2…. Find P_{110} if:

(a) $P_0 = 0$, and

(b) $P_{n+1} = P_n + n + (n+1)$, for $n \geq 0$

This student calculated some terms by hand and recognized that P_n are the square numbers. She demonstrated it as in Figure 2, which is very like Figure 1(B): in each case, one was added to each (vertical) "group" in the original situation (P_3), and another new "group" (of size n+1) was added to get to the new situation (P_4). While not all genotypes will be used in multiple situations, the "transfer" of this genotype between situations indicates that this genotype is a thing in itself.

It should be noted that it is often difficult to make these types of genotypic observations because it needs a *microgenetic* (Kuhn, 1995) approach to observations. While this is possible in research and in classrooms, it requires a different stance on the part of the observer. It also requires a different approach to formative assessment, one in which the teacher looks not only at what problems can be solved, but also looks in detail at *how* they are solved.

Multiplication in the Literature

Steffe (1994) is an example of what happens when one focuses only on creating a taxonomy of the various aspects of learning think multiplicatively. The components identified (uniting,

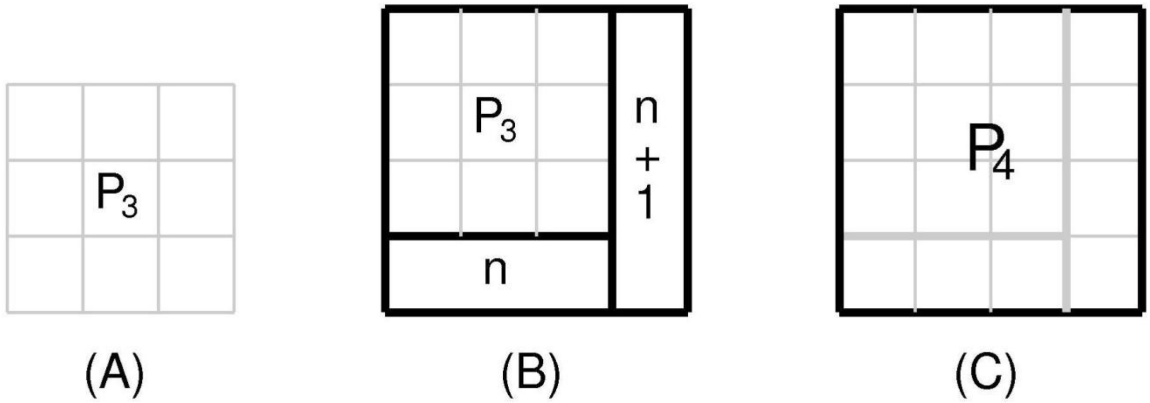

Figure 2. (A) P3. (B). Extension of P3 to P4. (C) Final explanation for P4

iterating composite units, etc.) are a step toward identifying what, in the GLP metaphor, constitute the genetic material of the thinking involved, but these do not address the way this thinking evolves or comes into being. While we need these classifications, they do little to move any student forward in his or her thinking without some concept for how new ideas can grow and develop. One of Steffe's subjects, Tyrone, provides a powerful example for the way thinking works in our metaphor. Tyrone clearly has a sense of multiplication as repeated addition, and demonstrates this through his initial step-counting by units of 20 to determine 20 × 20. When asked to calculate 30 × 20, he starts from his previous answer to 20 × 20 and step-counts for another 10 groups of 20. The numbers he has encountered allow him to use an existing strategy (step-counting) to solve two very similar problems; he clearly understands that 30 × 20 = (20 + 10) × 20 = 20 × 20 + 10 × 20 on an intuitive level. Tyrone is demonstrating problem solving abilities that are inherent in the ability to mutate an existing solution strategy. Because this was successful, Tyrone will now be encouraged to continue finding ways to adapt this strategy.

Although all new student ideas are formed from prior understandings, like natural evolution, GLP does not always produce successful strategies. The fossil record shows events, like the Cambrian Explosion (Gould, 1990), that have left their mark in the Burgess Shale, showing a huge variety of evolutionary solutions, none of which are found in today's natural world. The same is true for GLP. Van Dooren, deBock and Verschaffel (2010) provide copious examples of students hybridizing existing strategies to produce new strategies that are then casually dropped. Their study looked at how 3^{rd} – 5^{th} graders handled problems that were either additive or multiplicative in structure, having either integer ratios or non-integer ratios. Younger students relied on additive strategies for all problem types, while the students in the later years used methods more appropriate to the problem. In the intervening years, however, the students tended to use strategies that were genetic hybridizations of previous approaches, trying multiple strategies and adapting as needed to deal with the specific numbers in a problem.

Agostino, Johnson, and Pascual-Leone (2010) studied whether children approach different types of single-step and multiple-step multiplicative reasoning problems (scalar, array, combinatorial, or proportion) with different strategies. The researchers applied cognitive psychology principles that describe the general information processing skills being used as either related to inhibition, updating, shifting or mental attention capacity. While these are largely internal processes, relating to how the various executive functions are used, inhibition is related to a person's ability to stifle immediate responses to environmental cues in order to provide the other functions time to operate. All four components have been shown to relate to mathematical ability. Shifting refers to one's ability to jump easily between sets of information or problem tasks; this ability seems crucial if one is to enact a hybridization of existing strategies. More significantly, the authors show that age mediates the degree to which all of four factors play a role in problem solving, showing the evolution of children's multiplicative thinking over time.

Throughout GLP, the environment plays a crucial role in providing problem solvers access to the problem, access to other ideas about the problem and feedback about their solutions. This can come in the form of interpersonal communication, as it did for Joy (Trowell, 2012) where such experiences help the problem solver shift from "getting an answer" to "learning how to solve problems." Joy describes her process interviews, referring to changing strategies and falling back on previous approaches just to try something different, demonstrating clearly her attempts to mutate and hybridize strategies. Her view of mathematics and problem solving as very personal is a clear expression of the unique evolutionary path that each learner will generate in the absence of a single, universal law of what constitutes a "best solution strategy" similar to the way organisms develop uniquely from one another based on countless contingent events in evolutionary time.

Conclusions

A genetic learning approach, then, provides insight into the dynamics of learning, and hence can guide teachers' approaches to formative assessment and designing classroom learning environments. Taking a GLP approach to formative assessment requires teachers to look beyond the usual phenotypic assessments and to gather information about the underlying genotypes which are not always easily observed. By looking at students' intellectual genotypes during formative assessment, and thinking about what combinations of genetic material might be fruitful to promote, new ways of dealing with student understandings and problems, and new ways of supporting learning may be possible. A GLP approach to teaching also forces teachers to think of learning environments not merely as places where new material can be learned, but to explicitly promote possible combinations of existing student ideas.

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SUPPORT FOR STUDENTS LEARNING MATHEMATICS VIA STUDENT-CENTERED CURRICULA

Hannah Slovin
University of Hawaiʻi at Mānoa
hslovin@hawaii.edu

Fay Zenigami
University of Hawaiʻi at Mānoa
zenigami@hawaii.edu

Kavita Rao
University of Hawaiʻi at Mānoa
kavitar@hawaii.edu

Rhonda Black
University of Hawaiʻi at Mānoa
rblack@hawaii.edu

We conducted research on supports needed for at-risk students learning mathematics through curricula developed for a constructivist environment. Classroom observation data from a first grade class and a sixth grade class identified areas where the respective curriculum and pedagogy promoting student-centered learning posed specific challenges for struggling learners and suggested the potential supports that could help students access critical content and processes. Mathematics education faculty and special education faculty from the University of Hawaiʻi, College of Education collaborated on this study.

What does research say about “best practice” for students to learn mathematics with understanding? What does it say about “best practice” for a struggling student or one with special needs placed in a general education mathematics class? While we have moved to include students with varying needs into the same general education classes, the research traditions guiding us in that endeavor do not necessarily recommend the same strategies for the general student population and those newly integrated students to be successful. We sought to identify elements of two reform-based curricula that afforded at-risk and struggling students opportunities to be successful learners in mathematics class and to use the results to make recommendations to the classroom teachers for further supports.

Literature Review

The equity principle in the NCTM *Principles and Standards for School Mathematics* (PSSM) (NCTM, 2000) calls for all students to have opportunities to access challenging curriculum “regardless of personal characteristics, backgrounds, or physical challenges” (p. 12). In addition, legislative measures, such as the Individuals with Disabilities Education Improvement Act (IDEA, 2004) require that all students be exposed to the same curriculum and assessed by the same means. The task has fallen to classroom teachers—both general education and special education teachers—to provide a substantive mathematics education for all students but with little guidance on how best to accomplish the charge. One reason for the paltry support for teachers is the discrepancy between the numbers of research studies on students' mathematics disabilities versus students' reading disabilities. From 1996 to 2005, the ratio of studies focusing on reading versus those focused on mathematics was 14:1 (622 reading disability studies to 43 mathematics disability studies) (Gersten, Clarke, & Mazzocco, 2007). With so little direction from research, teachers must do the best they can to meet the needs of all students.

One model suggested to support students with special needs in general mathematics classes is collaborative teaching (Boyd & Bargerhuff, 2009; Sileo & van Garderen, 2010; van Garderen, Scheurmann, Jackson & Hampton, 2009). This model relies on the mathematics teacher and the special education teacher each contributing their unique knowledge and expertise to ensure students' success. The mathematics teacher's knowledge of the curriculum and content and the special education teacher's knowledge of effective intervention strategies work together to maximize students' learning.

But as van Garderen and her colleagues (2009) point out, this collaboration is "easier said than done" (p. 57). With little guidance, tension can arise between the mathematics teacher and the special education teacher whose approaches to practice can stem from different and often contradictory philosophies and theoretical foundations. "Best practice" from the mathematics education perspective envisions students engaged in explorations of new ideas posed by non-traditional problems that have multiple solution paths and often multiple solutions. Classroom interactions are built on social constructivist theories in which students negotiate meanings and establish understandings through activity and discussion with peers. By contrast, for the special education teacher, "best practice" means using students' individualized strengths and weaknesses to set learning goals that emphasize specific, measurable objectives geared mostly to developing procedural accuracy and fluency. Special education teachers use an instructivist approach to teaching, including giving clear, unambiguous directions, modeling how to carry out a procedure and allowing sufficient time for guided practice (Boyd & Bargerhuff, 2009).

Education researchers interested in equity matters for students with diverse cultural backgrounds (e.g. Delpit, 1988; Parks, 2010) have also suggested that reform methods may not advantage all children, particularly with respect to the dynamics of classroom interactions. Some of their concerns can be applied to at-risk and struggling learners. However, Delpit (1988) does not subscribe to the debate over which approach is better, process-focused reform programs or skill-focused traditional programs, saying the dichotomy is "false" (p. 296). She does, however, propose an alternative where students engage in content that builds creative problem solving and sense making and also learn the explicit skills needed to be successful in such an environment. Researchers in mathematics education have just begun to explore what this middle ground might look like. In particular, Boaler (2002) has suggested that the fact that some reform practices may disadvantage some students does not necessarily mean that traditional practices would provide more learning opportunities for these same students. Instead of labeling the curriculum as inappropriate or claiming there is something lacking in the students, Boaler (2002) suggests that teachers need to be supported in helping students acquire learning processes.

Methodology

The project was conducted at a public charter school that partners with the Curriculum Research & Development Group (CRDG) at the University of Hawai‘i College of Education. Students are admitted through a stratified random selection process from a pool of applicants to represent the broader public school population of the State ethnically and racially, socio-economically, and with respect to prior achievement. We report on observations in grade 1 (n = 10)[*] and one section of grade 6 (n = 26). The students have mathematics lessons every day, with the typical mathematics session for all grade levels lasting 45 minutes.

Measure Up (MU) is a K–5 program used at the charter school that introduces topics through the context of continuous measure, enabling young students to reason algebraically about relationships and build an understanding of mathematical structures (Slovin, Okazaki, Venenciano & Zenigami, 2007). In grades 6 and 7, the Reshaping Mathematics for Understanding (RMU) curriculum starts from a geometric approach and moves to an algebraic perspective as students progress through middle grades topics. The charter school uses both programs with their students, exclusively. Both curricula were developed at CRDG and use problem solving to introduce new concepts and develop understanding starting from a conceptual level and moving to a skill level over time. Tasks in both programs are designed such that children of diverse abilities are able to access and respond to challenging mathematics problems.

The project design comprised three phases. In the first phase, before observing mathematics classes, team members reviewed the curricula. The intent of this examination was to give special education faculty team members an opportunity to preview the content and tasks students are

[*] Grades K – 5 in this charter school have 10 students in each grade. Students are grouped in K-1, 2-3, and 4-5 grade spans except for mathematics, which is single grade taught.

expected to learn and the inquiry approach used to present them. The team developed an initial observation protocol to pilot. The protocol, based on UDL guidelines (http://www.udlcenter.org/), was designed to enable the team to identify which components of the existing curricula address the needs of struggling students, including those receiving special education services.

One mathematics education faculty member and one special education faculty member were on each observation team. Student materials were provided to observers. There was some variation in how the protocol was used during the observation. The mathematics education team members less familiar with the UDL framework did not take observation notes directly into the cells of the framework. Rather, these observers transferred their notes into the framework after the observation, with the intent of matching notes to cell descriptors.

The second phase of the project started with team discussions about the outcomes of observations in the first phase. Team members noted that the UDL framework narrowed the focus of the observations and were concerned that the pre-determined categories of the protocol could limit what was recorded, and thus, potentially important aspects of the classroom events might not emerge. Since this was a new collaboration, and one across areas of scholarship, it was important that the focus not be restricted prematurely. The revised observation plan and protocol was semi-structured, allowing researchers to develop a broader understanding of the classrooms before targeting particular aspects of the mathematics classes related to how teachers use the existing curricula to meet the needs of struggling students.

The classroom observations during phase two were of two types. Early in the school year, classrooms were observed holistically using the revised protocol that enabled the researchers to study interventions enacted through the interactions among materials, teachers, and learners (Slovin, 2010). Three two-consecutive-day observation cycles were conducted in the first two months of the school year. Observing on two consecutive days enabled team members to follow students' development of a topic over time better than observing once a week. Grounded theory (Glaser & Strauss, 1967; Strauss, A., & Corbin, J., 1998) was used to analyze the data, looking for emerging themes that indicated affordances at a whole-group level. The second type of observation was conducted in the spring semester and involved several case studies on the application of identified instructional strategies to individual struggling students.

The third phase of the project was based on the outcomes of the first two phases. Recommendations to the existing curricula to support mathematics learning and teaching with diverse learners were made so that teachers can use the existing curricula to meet the needs of struggling students.

Findings

The main objectives of the project were (a) to examine the affordances provided within the existing curriculum and pedagogy and (b) to discuss and recommend further supports that could be put in place at the whole class level and for supplementary instruction that the struggling learners were receiving.

A common feature of the elementary and middle school curricula in the observed classes is that students were often engaged in solving problems having multiple solutions, including those with more than one outcome as well as those with more than one approach to reaching an answer. Furthermore, students in both classes were expected to collaborate and engage in experiences from which conceptual understanding was developed. Utilizing different strategies, comparing and contrasting methods, and making connections between approaches allowed students more access to solve and make sense of problems and provided opportunities to make mathematical generalizations. Communicating through concrete, pictorial, oral, and written forms required students to transmit, receive and reflect on shared information to begin developing their own understandings of the mathematics.

In the first grade class, where the curriculum is designed for instruction to be more student-centered rather than teacher-centered, supports were often aimed at helping students learn to be learners. The teacher guided students, both implicitly and explicitly, on how to work with others, how to reason, how to ask questions, how to discuss with a partner, how to speak in front of the class, how to listen thoughtfully, and how to see from multiple points of view through the context of the problems they were solving.

During one observation, the first graders were paired to work on a task involving a spring scale to develop their understanding of unit and, with their partner, were first directed to explore how the spring scale worked. One student, identified as a subject of the Phase 2 case study observation, initially took possession of the materials and began exploring independently. The teacher, noticing this behavior among other pairs as well, reiterated that students were to share the material with their partners. The observed student then gave the spring scale to his partner and began to shout out some of his observations. The teacher reminded him individually that students were expected to discuss the observations with their partners. The student then spoke to his partner about his observations, took her hand when she agreed with him, and raised them together to indicate they were ready to share the observations with the rest of the class.

Next, the pairs were provided several sets of mass-units. The observed student declared that he would place the first three mass-units and that his partner would mark and label each unit on the spring scale after he placed them; then they would switch roles for the next three mass-units. This behavior suggested that he had begun to apply the notion of shared responsibility in the learning process.

In discussions with the classroom teacher following the observations, it was agreed that an inquiry approach should also be used in any additional instruction with the struggling students and that direct methods would not be a suitable practice to support learning with the curriculum. While one-to-one work in a tutoring situation might increase student accountability, having small groups might provide better opportunity for students to gain insights from each other, which, in turn, would model the whole class experience.

In grade 6, although students had more school experience, navigating the complexities of learning critical mathematics content and processes in a constructivist environment was new for many students. From our whole-class observations in grade 6, we derived themes that we termed "dilemmas" (Lampert, 1985), meaning that these features of the RMU content and pedagogy provided struggling learners with both affordances and challenges.

1. Conceptual development: There is a broad range in students' conceptual development in the mathematics topics. Class discussions were enriched by this diversity, often producing varying points of view. When following a wide-ranging discussion became challenging for some students to follow, the teacher had to intervene to help manage the discussion.
2. Rich content: The concepts are complex (e.g., the many properties of transformations) and several concepts may be discussed concurrently to foster connections among them. Making connections among concepts and topics promoted opportunities for students to achieve deeper understanding and to think critically. Yet some students may have had difficulty seeing how concepts are related, making it necessary for the teacher to guide their thinking to consider links between the ideas.
3. Verbalization skills: Students use different terms to describe mathematical ideas. Multiple forms of representing ideas afforded students greater participation in the mathematical

discussion. The challenge was for the teacher and students to negotiate a shared meaning among the various expressions of ideas.

4. Multiple points of view: Students use multiple methods to solve problems, giving multiple explanations and rationales. Problems designed to promote different solution strategies afforded access to a wider range of students. It may have been difficult for some students and the teacher to express and understand the thought processes used in the multiple approaches.
5. Fluid protocol: With student-centered discourse, the direction of the discussion may change to accommodate students' needs. Allowing students to introduce new ideas and orchestrating a discussion to address students' questions and uncertainties gave students more opportunities to develop understanding. This feature often required the teacher to make a judgment about when a discussion needed redirecting.
6. Learning goal: The over-riding goals are for students to develop deep understanding and to make sense of the mathematics. Expecting all students to go beyond acquiring skills to produce correct answers necessitated many teacher decisions regarding managing class discourse and program pacing.

In both the grade 1 and grade 6 classes, students required teacher support specific to helping them develop the skills and processes critical to learning in a student-centered environment. This is significant because our professional discussion has so far taken an either-or approach, either direct or open-ended tasks and discussions. This small study has demonstrated that teachers' pedagogical strategies can be explicitly designed to support learning in an environment where children actively construct meaning and build understanding. Moreover, we believe that teacher education, both pre-service and in-service, should begin to focus on the instructional strategies teachers need to be prepared to make learning challenging but accessible for all students.

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HOW THE HAND MIRRORS THE MIND: THE EMBODIMENT OF NUMERICAL COGNITION

Thomas J. Faulkenberry
Texas A&M University - Commerce
Thomas.Faulkenberry@tamuc.edu

Much knowledge about how the mind does mathematics is based on the traditional, computer-based metaphor of cognition that assumes cognition is stage-based and independent of the motor cortex. In the present study, I provide evidence for an alternative view. I recorded participants' hand movements as they chose the correct parity (odd/even) for single-digit numerals. Distributional analyses of these movements indicated that responses resulted from competition between parallel and partially-active mental representations rather than occurring in discrete stages. Furthermore, this competition was carried through to the motor cortex, indicating that numerical representations are more tied to bodily affordances than previously thought.

Researchers have been investigating mathematics learning for many years, particularly through the paradigm of cognitive psychology. From early attempts to understand how arithmetic facts are organized (Ashcraft & Battaglia, 1978) to formal models that specify the various cognitive processes involved in mathematical problem solving (Anderson, 2005), most of these studies have made the implicit (albeit, metaphorical) assumption that the mind operates like a computer. That is, perception informs cognition, and cognition informs action. In this view, higher-level cognitive systems (memory, executive control, etc.) are thought to be quite separate from the lower-level systems (perception, motor action).

This modular, discrete-systems approach to cognition has been the fundamental underpinning of most of our understanding of how the mind does mathematics. From the point of view of "mathematics is a collection of abstract ideas," it makes sense that mathematical objects could be learned and operated on in a purely abstract fashion without any interaction with other (non-cognitive) neural systems, such as the motor cortex. However, Lakoff and Nunez (2000) proposed the hypothesis that mathematics is learned through conceptual metaphor, a mechanism for converting embodied (sensori-motor) reasoning to abstract reasoning. At the time, unfortunately, their view was almost entirely philosophical, and it elicited much debate between cognitive scientists and mathematicians. Without behavioral evidence, the debate would be sure to stay within the realm of philosophy, and as such, not be widely accepted among mathematicians and psychologists alike.

In recent years, however, other cognitive scientists have proposed a view similar to that of

Lakoff and Nunez: specifically, that the human mind is not a modular computer, but rather a rich, dynamic system of parallel and partially-active representations (Spivey, 2007). In this view, decisions are not made through modular "switches," but instead are the result of competition among many different partially-activated responses, simultaneously informed by feedback from many other systems, including (especially) the motor systems.

The canonical example of this view is found in the language-processing literature (Spivey, Grosjean, & Knoblich, 2005). In a language-comprehension task, they asked participants to listen to words and, with a computer mouse, choose the picture that correctly represented the spoken word. During this task, they measured participants' hand positions by continuously recording the (x,y) coordinates of their mouse. They found that when words were phonetically similar (CANDY versus CANDLE, see Figure 1), the mouse tracks tended to deviate toward the incorrect alternative early in the response process, but eventually settle in to the correct answer. This is commonly taken as evidence for an embodied view of cognition, where responses result from a dynamic competition between partially-active, unstable mental representations. In the classic, modular view of cognition, the hand positions would not be so sensitive to influence from the decision process, as the motor system would not be called upon until the decision was made in the language center of the brain.

Until now, no studies have investigated the processing of numerical information within such a continuous, embodied-cognition framework. This is unfortunate, as the work of Lakoff and Nunez (2000) has set the stage for such research to tease apart the contributions of different cognitive and perceptual systems to numerical cognition. In the present study, I used the hand-tracking paradigm of Spivey (2007) to capture the formation of numerical representations during a parity judgment task. Participants quickly judged whether single digit numbers were even or odd. Responses were either consistent with spatial orientation of numbers (i.e., small numbers on left side or large numbers on right side) or inconsistent (i.e., small numbers on right side, large numbers on left side. Two competing predictions were then tested. If numerical cognition is indeed part of an embodied system, then hand trajectories in the *inconsistent* condition should show a pull in the direction of the incorrect alternative, reflecting a settling of partial activations of response alternatives during the response. If, on the other hand, numerical cognition is

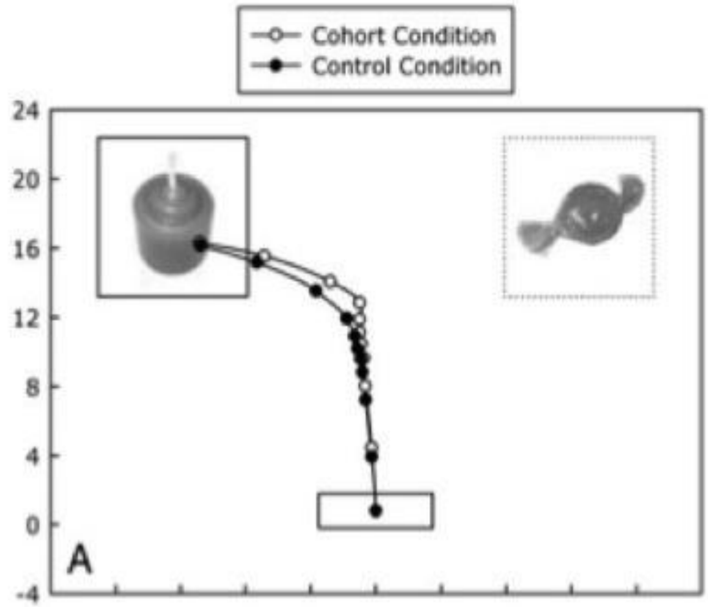

Figure 4: Words that are phonological similar (candy versus candle) show a competition throughout the response.

modular and stage-based, then we should see little attraction toward the incorrect alternatives, since the incompatibility would be resolved before the motor output stage.

Method

Participants

45 undergraduate students (35 female, mean age 24.3 years) participated in the present study.

Stimuli and Procedure

Single digit numerals (excluding 5, as is common in the numerical processing literature) were presented on a computer screen using the software package MouseTracker (Freeman & Ambady, 2010). Participants were told that, on every trial, a number would appear in the center of the screen, and they would be asked to choose, as quickly as possible, whether the number was even or odd. After participants clicked a “Start” button centered at the bottom of the screen, response labels “Even” and “Odd” appeared at the top left and right of the screen (the order of these labels was switched once midway through the experiment). Participants then clicked on the correct of these two options; while doing this, I recorded the streaming (x,y) coordinates of the computer mouse approximately 70 times per second. Each participant completed 640 trials. This yielded a rich data set of hand trajectories, which in the spirit of Spivey and colleagues (2005) directly reflects the mental processes that occurred during the numerical decision making process.

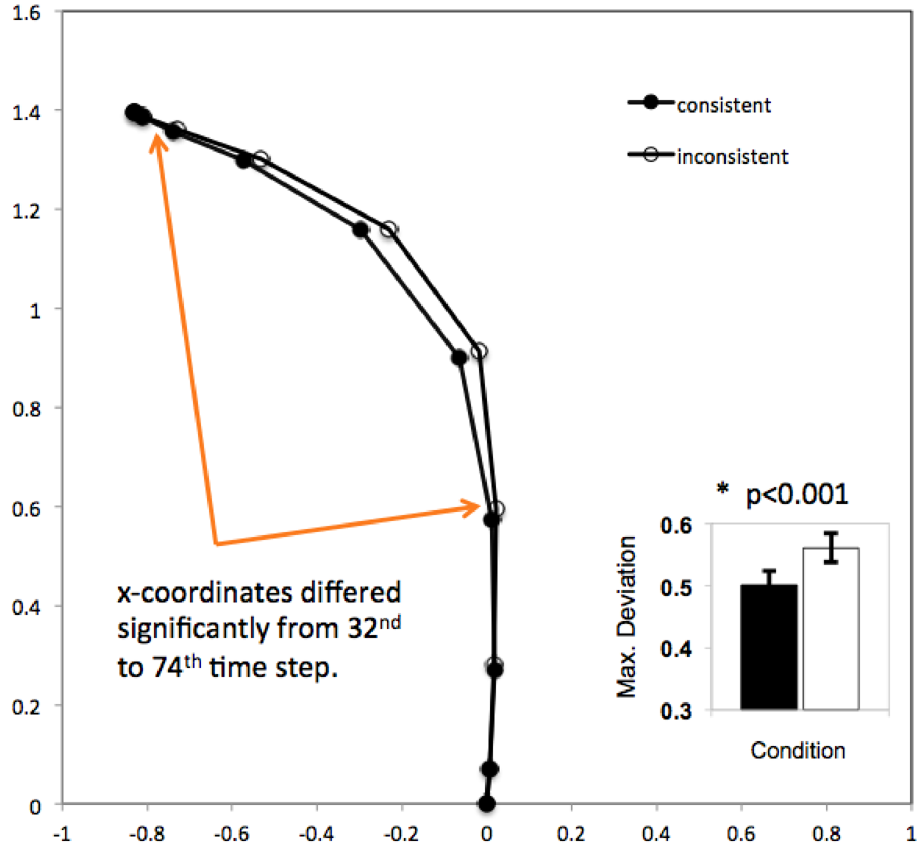

Figure 5: Average hand trajectories during the numerical parity task, separated as a function of spatial compatibility.

Results and Discussion

All hand trajectories were rescaled into a standard coordinate space, [-1,1] x [0,1.5]. To analyze movements independent of reaction times, I used linear interpolation to normalize all trajectories to consist of 101 times steps. This is important so that trajectories of differing time scales can be averaged over multiple trials. In addition, for ease of visualization, all trajectories for responses on the right-hand side of the screen were reflected to the left side of the screen.

The first analysis is with respect to the hand trajectories in each of two spatial compatibility conditions. On *consistent* trials, participants responded to small numbers (1,2,3,4) on the left side of the screen and large numbers (5,6,7,8) on the right side of the screen. On *inconsistent* trials, these were reversed. These conditions are motivated by the robust finding that most English-speaking adults have an implicit left-right number orientation (Dehaene, Bossini, & Giroux, 1993).

To analyze the hand trajectories, I computed an average trajectory across all participants for each of the two spatial compatibility conditions. As can be seen in Figure 2, hand trajectories in the inconsistent condition are a bit “pulled away” from the trajectories for the consistent condition. One interpretation of this is that the trajectories in the inconsistent condition continuously attracted toward the incorrect response alternative throughout much of the response, indicating a high degree of competition between the two response alternatives. Indeed, across all trials, the average trajectory was significantly closer in proximity to the incorrect response alternative from the 32^{nd} to the 74^{th} time step.

For a trial-by-trial index of the degree to which the incorrect response alternative was partially active, I computed the *maximum deviation:* the largest positive x-coordinate deviation from an ideal response trajectory (a straight line between the start button and the response) for each of the 101 time steps. As indexed by maximum deviation, trajectories for inconsistent responses ($M=0.56$, $SE=0.02$) were significantly more attracted to the incorrect response alternative, compared with trajectories for consistent responses ($M=0.50$, $SE=0.02$), $t(44)=6.41$, $p < 0.0001$.

Across both measures, the data reflect that during the numerical decision process, participants formed partially-active representations of both response alternatives until the “winning” representation was stabilized and the correct answer was chosen. Initially, this seems to support the embodied view of cognition. However, an alternative explanation could instead explain the data. It could be the case that the smooth, continuous attraction we are seeing is the result of averaging across trials. For example, if some trials showed zero attraction (i.e., the participants' hands moved directly toward the correct answer) and other trials were sharply deflected midflight after realization of an error, the appearance of the average trajectories would be smooth even though the cognitive processes involved were modular (that is, motor responses were not initiated until the decision was made). In this case, if we were to look at the distribution of the maximum deviation values, it would be distinctly bimodal; simply put, some of the values would be small (indicating direct trajectories) and others would be large (reflecting the midflight correction of an almost incorrect response).

To test whether this is the case, I performed a distributional analysis on the collection of maximum deviation values across all trials. Each of the 28,800 values (640 values for each of 45 participants) was converted to a *z*-score. Figure 3 depicts the distribution of these maximum

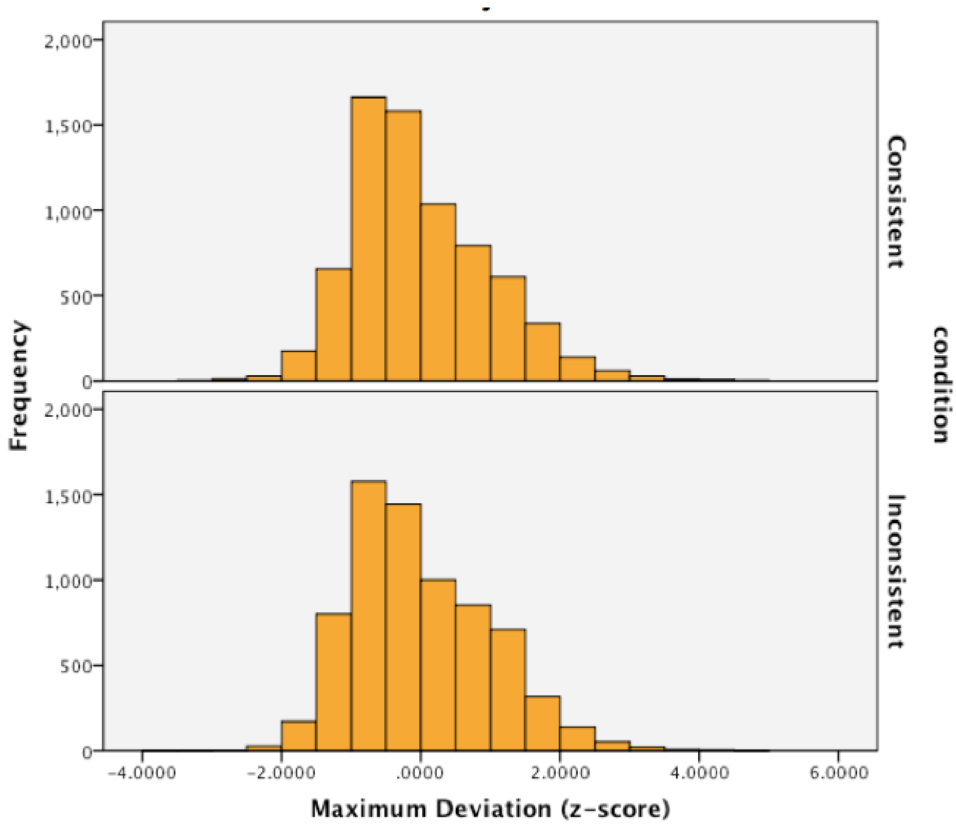

Figure 6: Distributions of maximum deviation values for consistent and inconsistent trials

deviation values for both consistent and inconsistent trials. Notice, critically, that the inconsistent trials do not differ in shape from the consistent trials, nor do they appear bimodal. Modality analysis confirms that the distribution of inconsistent trials is indeed not bimodal: the computed coefficient of bimodality was 0.423, with $b > 0.555$ representing the minimum value for a distribution to be considered bimodal. Also, a Kolmogorov-Smirnov test confirms that the distribution of values on consistent trials is not significantly different from those in inconsistent trials ($z = 1.33$, $p > 0.06$). These data indicate that the distribution of maximum deviation values is not bimodal, and that the smooth, continuous attraction away from the correct answer in the inconsistent trials is *not* the result of participants' quickly correcting their fast, incorrect initial responses.

In summary, we found an interesting pattern of responses when people are making quick judgments about the parity of a number (whether it is even or odd). Particularly, the size of the number affects the dynamics of our hand responses (even though the size is irrelevant to the task). This effect was captured by looking at the streaming path of computer mouse coordinates as participants selected the correct response label (even or odd). There seemed to be an automatic activation of numerical size that was carried out in participants hand movements (see Figure 2). This directly supports the hypothesis of embodied cognition. However, this pattern could have also resulted from an "averaging" across trials; on inconsistent trials, the mouse movement could have initially been in the wrong direction, then immediately corrected midflight to the correct response alternative. However, an trial-by-trial analysis rules this possibility out (see Figure 3).

Together, these results comprise an important first step in establishing the embodiment of numerical cognition. From the work of Lakoff and Nunez (2000), an important philosophical claim was made: mathematics is entirely the creation of humans using entirely human qualities. In other words, mathematics as we know it could not have been "invented" without the bodily affordances that make us human. While this claim may seem more the realm of philosophers and science fiction writers, the present results provide some evidence that even numerical decisions are intimately tied to our bodily states.

General Discussion

The results of the present study indicate that numerical processing does not take place independently from our bodily states. Specifically, we found that when tracking hand movements in even the most simple task (a parity judgment task), the hand movements reflected a continuous, dynamic system of partially activated cognitive states that would not be possible in the traditional, computer-based metaphor of mind.

At first, it may be difficult to see how these results relate to discussions in mathematics education. Indeed, the results from such research are valuable to mathematics educators because, together, they lend theoretical support to the idea of embodied mathematics. Embodied mathematics is the view that mathematics is not completely an abstract, or "pure," discipline, but rather is the product of a conceptual system that is ultimately grounded in bodily states. In a sense, this "humanizes" mathematics. This kind of evidence also tells us that since mathematics is tied to our body-grounded conceptual systems, it should be taught from that point of view. That is, as Nunez and colleagues put it:

> "Students (and teachers) should know that mathematical theorems, proofs, and objects are about ideas, and that these ideas are situated and meaningful because they are grounded in our bodily experience as social animals." (Nunez, Edwards, & Matos, 1999, p. 62).

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DEVELOPING DISCOURSE THAT PROMOTES REASONING AND PROOF: A CASE STUDY OF A CHINESE EXAMPLARY LESSON

Lianfang Lu
University Of Arkansas At Little Rock
Lxlu@Ualr.Edu

Thomas E. Ricks
Louisiana State University
Tomricks@Lsu.Edu

As part of a larger study that investigated mathematics teaching reform in China (Lu, 2011), this study focused on a high-quality model lesson that represented the recommended instructional practices in current Chinese mathematics education reform. We focused our analysis on the design of the lesson, the unfolding of discourse, and the development of students' mathematical reasoning and proof in the lesson.

Theoretical perspectives

Reasoning and proof are regarded as fundamental aspects in mathematics study for all grades (NCTM, 2000); however, research has shown that students have various difficulties when they were engaged in reasoning and proofs. For example, students tend to rely on the authority for truth; lack of understanding of proof; misuse empirical examples as arguments for proofs and lack of strategies of proofs (Harel& Sowder, 2007, Stylianou, Blanton, &Knuth, 2009; Thompson, Senk, Johnson, 2012).

Mathematics educators have noticed the role of discourse in helping students overcome these difficulties. A significant challenge teachers face is how to develop discourse that promotes robust mathematical understanding and abilities of reasoning (Stein, et al., 2008). Stein et al. (2000) highlighted that different tasks may engage students in different levels of mathematical reasoning. They pointed out that tasks in discourse with higher-level cognitive demands, such as *doing mathematics* activities provide opportunities to engage students in high-level thinking. Research also indicated that heuristic questions and suggestions asking for explanation and extension deepen students' reasoning (Goldin, 1997). For instance, Martino and Maher (1999) found that the teacher's posing of timely questions, while challenging learners to advance their understanding assist students find successful solutions. Yackel and Cobb (1996) identified sociomathematical norms that encourage listening and explanations support students to develop explanation, justifications, and mutual understanding. More recent interest has pointed to "inquiry-based" instructional approach to improve reasoning and proof through engaging students in problem-solving (Smith, 2012).

Social system perspectives view the classroom as a learning system in which the teacher and students are participants of a mathematics communication network. Social system perspectives may add insights for promoting deep and rich mathematical understanding and reasoning in discourse (Stein, et al., 2008). From social system perspectives, classroom discourse is not transmission of information but correlated and recursive action among classroom members (Fleener, 2002; Luhmann, 1999; Maturana &Varela, 1987; Sfard, 2008). The generation of meaning is not located inside the mind of individuals, but in the discourse itself. Discourse grows with the contributions from individuals' participation in the discourse, as Sfard highlighted,

Discourse rules of the mathematics classroom, rather than being implicitly dictated by the teacher through her own discursive actions, are an evolving product of the teacher's and students' collaboration efforts. (p. 202)

Such a perspective emphasizes a co-growth and a recursive process of developing reasoning and proof. It suggests that successful classroom mathematics discourse is self-generating, self-adapting, and self-maintaining; additionally, both participants and discourse cycles co-adapt and co-evolve. Sfard pointed out the richness and depth of understandings emerged in classroom discourse (the network of communications) reflect the quality of the discourse. Higher quality mathematics instruction (usually conducted by more experienced teachers) manifests richer discourse.

Data Sources and Methods

This study investigated how a recognized expert mathematics teacher in urban China designed and facilitated classroom discourse that promoted meaningful understanding and reasoning in a mathematics lesson focusing on the minimum criteria for determining if quadrilaterals are congruent (extending Side-Angle-Side and related postulates). The teacher, Mr. Wu (pseudonym) was both recognized as an outstanding mathematics teacher in the local city and had participated in the development of China's new national mathematics curriculum. Mr. Wu taught his exemplary model lesson to a random group of ninth graders during a district-level teaching development project as about 20 mathematics teachers from the district observed. The lesson lasted about one hour. Afterward, Mr. Wu held a debriefing session with the teachers. Data for this study included our observation and videotaping of the lesson and the debriefing session, and interviews with Mr. Wu and the director of the district-level teaching development project. Four video cameras were used to catch the teaching and learning from different angles.

One camera was fixed at the front to catch the general context of the whole teaching process. Another camera was fixed at the back of the class to catch the activities near the blackboard. The third and fourth were carried by the two researchers to capture classroom discussion, which primary focal point was the teacher. Documents from Mr. Wu's former presentations and several video clips of his previous teaching were also used. Data were discussed among the researchers. Case study methods were used for data analysis (Huberman & Miles, 1994).

Results

Findings indicates a recursive inquiry-based instructional design that emphasized the processes of conjecturing, testing, revising, and proof; evolving discourse with increasing mathematical understanding and reasoning skills was also documented. In particular, we traced the discourse throughout the lesson to understand the advancement of students' mathematical thinking and the strategies employed by the expert teacher in designing and developing rich and dynamic discourse.

The Task

In the lesson, the main learning task was to investigate under what conditions two quadrilaterals are congruent. The cognitive demand of the problem was "*doing mathematics*", which involved higher-level mathematical thinking (Stein et al., 2000). The task requires students to make conjectures and prove the conjecture. It allows multiple entrances to tackle the problem and embodies the essential skills of reasoning and different ways of proof. Mr. Wu indicated in the debriefing session that the objective of the lesson was to help students learn mathematical inquiry and to think mathematically. The designing of the task was aligned with the higher goal.

The Unfolding Discourse

The lesson began with an introduction activity to mathematical inquiry in which the teacher let students explore the ratio between the length and the width of a common sheet of size A4 paper. Mr. Wu initiated the conversation as below.

Wu: When I am looking at this piece of paper, there are a lot of questions I want to ask you. My first question is: have you ever paid attention to the length and width of it? What is the ratio between the length and width approximately? What do you think about it?

One student immediately said 3:2. Mr. Wu continually asked for other students' opinions. When seeing no more different responses, Mr. Wu posed an argument, "Why not 5:3?" He then urged students to think of some ways to test the conjectures. After students suggested measuring the paper, Mr. Wu and students measured and they agreed to change the ratio to something close to 1.5.

Wu: But there is always some error (in measuring). Like what I said before, a little less than 1.5. It was 1.4 something. So what do you think about the ratio is?

Mr. Wu continued leading students to make justifications and think beyond the empirical stage. The introductory activity gave students first-hand experience of the general process of inquiry: a recursive process of making conjectures, revising, and proving.

During the main section of the lesson, the teacher and students explored various possible situations of the problem. The teacher continued to question and promoted students to make conjectures and test and revise their conjectures, as illustrated in the following episode.

Mr. Wu started the main task with the question: "How do you determine if two polygons are congruent?" When students offered the definition of congruence of any polygons: "All the corresponding sides are equal, and also the angles," Mr. Wu challenged the answer with a series of questions.

Wu: Do you think the angles have to be the same? Aren't equal sides enough? Why not? I think it's perfect. If all the sides are equal, the two quadrilaterals are the same. What do you think?

Student: Suppose there is a quadrilateral, whose sides are fixed, but the quadrilateral is movable.

Wu: It is movable. Unstable. Can you give an example?

Student: A square and a rhombus.

Mr. Wu asked the student drew a picture on the black board. Then he re-voiced the student's explanation.

Wu: like a square and a rhombus have the same side length. That's what you meant, right? If a square and a rhombus have the same side length, their corresponding sides are equal, but they would not overlap completely like what he said before. So how can you correct that statement?

Figure 1 illustrates the dialogue of the lesson during the main session. As shown in Figure 1, the teacher initiated the main question "under what conditions two quadrilaterals are congruent", which was then developed into several sub-questions. The conversations resembled evolving chunks of unfolding mathematical understanding as the class began to establish concrete shared reference points from which to further develop their proof of the main question. It was centered on students’ explanation, constructing and justification and indicates a spiraling development around the main task. The conversation system determined the directions of discourse rather than an authority (Davis & Sumara, 2006; Sfard, 2008). The discourse became evolved and dynamic rather than convergent and linearly patterned. In the main session, students learned not only how to do mathematical inquiry, but also strategies, such as how to categorize situations and construct counter-examples. The lesson closed with the teacher and students applying and reflecting on those methods of inquiry.

Overall, the discourse demonstrated a recursive process of developing students’ mathematical thinking. At each stage, both the teacher and students made conjectures, revised, and proved the conjectures, leading to a higher level of mathematical thought. Figure 2 below captures the iterative and inquiry-based structure across the introduction, main session and closing. In the development of the discourse, Mr. Wu sometimes posed himself as someone who turned on students for solutions as he asked “what do we do?”, “what do you think?”. He showed not only his respect but also his passion to students’ solutions. Moreover, Mr. Wu was sensitive to student ideas and also flexible to adjust his teaching based on the dynamics of classroom interactions. When students’ answers were different from his expectation, instead of guiding students to what he was expecting, he made modifications of his instruction. His adjustment of his instruction was a comprehensive consideration of emerging ideas, students’ understanding, and learning objectives of the lesson. He was also able to help students develop their thinking while also lead the discussion back to his original plan in a later time. In the process, student ideas were a “generator of meaning” (Peressini & Knuth, 1998). Both the teacher and students were equal participants of the inquiry process. Overall, the features demonstrated in Mr. Wu’s lesson for developing the discourse include 1) a democratic environment to engage students in the investigation of the problem; 2) passion and appreciation for students’ ideas; 3) questions requiring explanation and justification; 4) arguments stimulating sense-making; 5) flexible instruction guided by a comprehensive consideration of emerging ideas, students’ understanding, and lesson objectives; 6)recursive conversation built on the co-evolvement of the teacher and students.

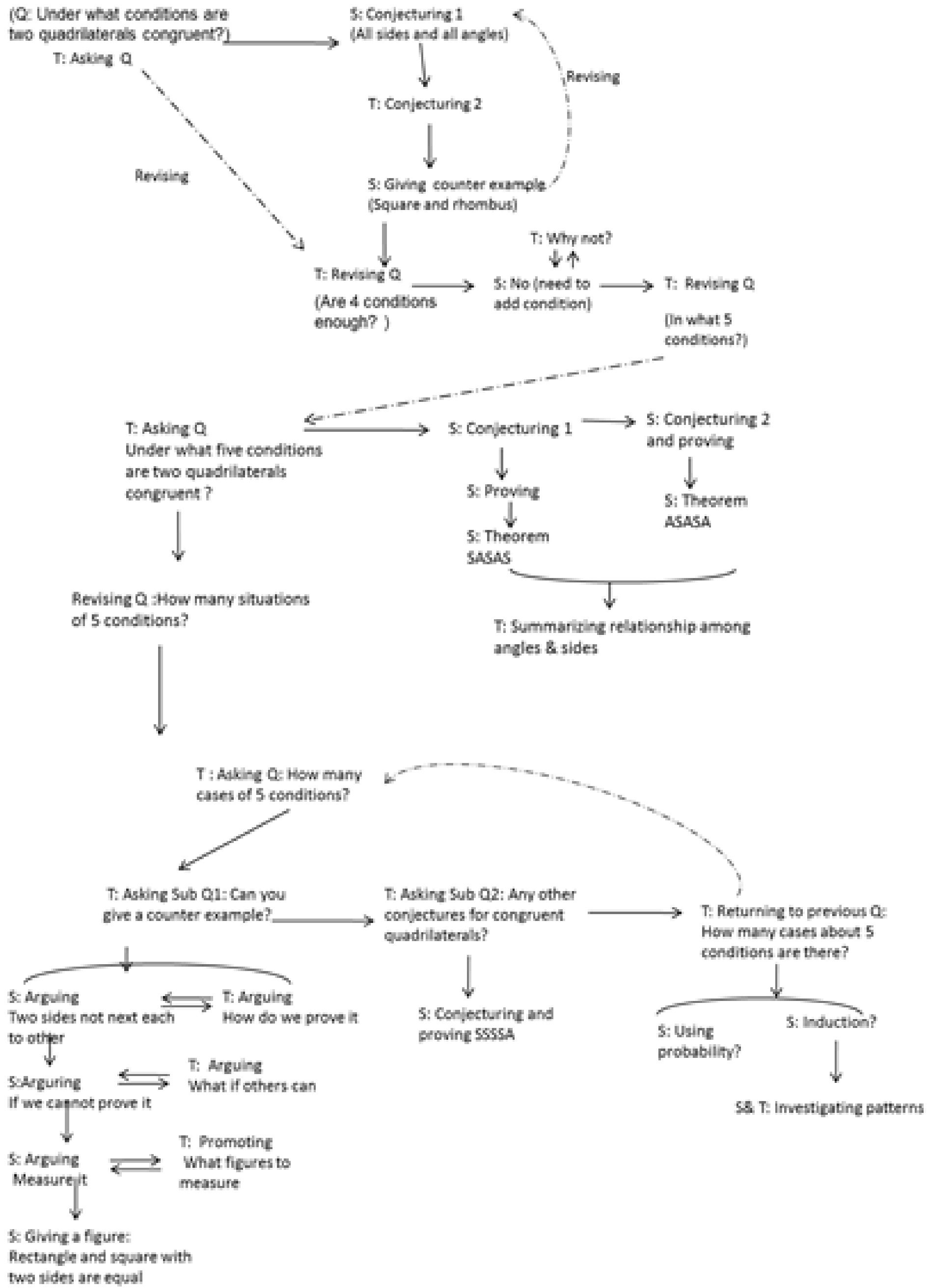

Figure 1: The Discourse Patterns During the Main Section of the Lesson

(T: the teacher, S: students Q: questions)

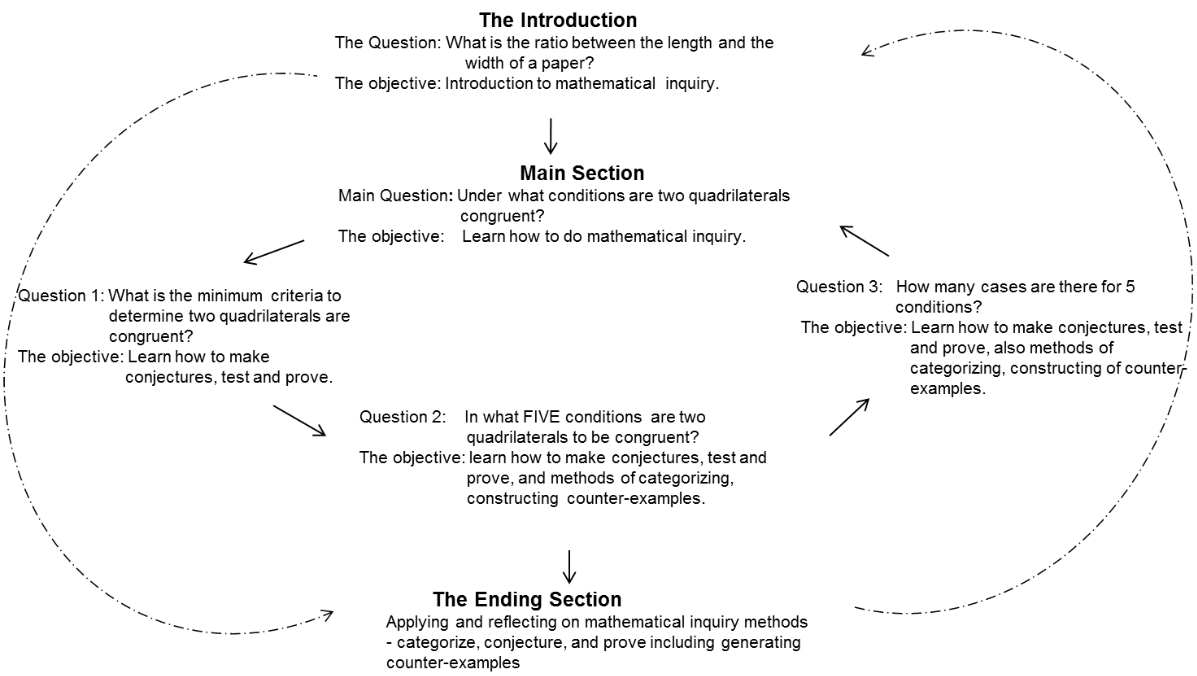

Figure 2: The Structure of the Lesson

Manifestations of Developing Student Understanding

In this lesson, the students collectively conjectured about the congruence of quadrilaterals using four possibilities:

1) Four sides and one angle: Side-Side-Side-Side-Angle (SSSSA)
2) Three sides and two angles: Side-Angle-Side-Angle-Side (SASAS) and Side-Side-Side-Angle-Angle (SSSAA)
3) Two sides and three angles: Side-Side-Angle-Angle-Angle, (SSAAA) and Side-Angle-Side-Angle-Angle-Angle (SASAA)
4) One side and four angles: Side-Angle-Angle-Angle-Angle (SAAAA)

Students then agreed that two quadrilaterals are congruent under SASAS, SSAAA, and SSSSA and proved these new theorems using common properties of triangle congruence (SSS, SAS, AAS). They also generated counter-examples to show when congruence of two quadrilaterals cannot be established (for example, SASAA, or a rectangle and a square with two equal sides). During the exploration of the criteria for the congruence of quadrilaterals, students also explored possible cases regarding the positions of the angles and sides in quadrilaterals. The

Figure 3 blow illustrates the developing of students' reasoning and proof. It also reveals the richness and depth of mathematical understanding and thinking.

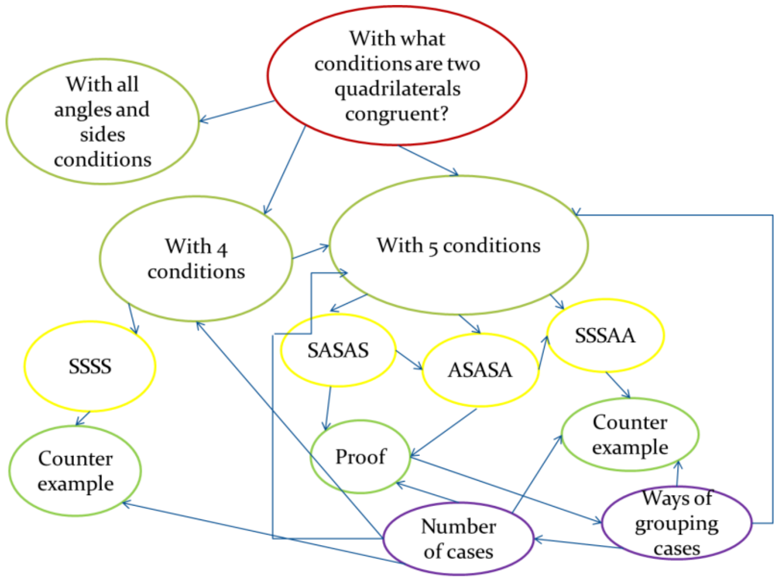

Figure 3 The Development of Reasoning and Proof

Educational Significance

Recent mathematics education reform has highlighted the importance of robust classroom discourse as critical for advancing reasoning and proof of students. In China, exemplary model lessons are often taught by expert teachers during professional development to illustrate these reform ideas and to suggest possible implementation approaches (Huang, Zhang & Li, 2011). These model lessons manifest successful classroom discourse as the teacher and students develop mathematical content jointly. Examination of this particular lesson by Mr. Wu provides insights and strategies for teachers to develop rich mathematics discourse that promotes reasoning and proof in US mathematics classrooms.

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